NOTES ON HOPF MODULES, INTEGRALS AND FROBENIUS HOPF ALGEBRAS.

PAOLO SARACCO

1. Hopf modules and the structure theorem

Let $\Bbbk$ be a commutative ring and denote by $\mathfrak{M}$ the category of $\Bbbk$-modules and their morphisms. In this section, B will be a bialgebra.

Definition 1.1. A *(right) Hopf module* over B is a $\Bbbk$-module M with a B-action $\mu : M \otimes B \to B : m \otimes b \mapsto m \cdot b$ and a B-coaction $\delta : M \to M \otimes B : m \mapsto \sum m_{[0]} \otimes m_{[1]}$ such that

$$\delta\left(m \cdot b\right) = \delta\left(m\right) \cdot b,$$

that is to say,

$$\sum \left(m \cdot b\right)_{[0]} \otimes \left(m \cdot b\right)_{[1]} = \sum m_{[0]} \cdot b_{(1)} \otimes m_{[1]} b_{(2)}$$

for all $m \in M$, $b \in B$. A morphism of Hopf modules is a B-linear, B-colinear map between them. The category of Hopf modules over B and their morphisms will be denoted by $\mathfrak{M}_B^B$.

Example 1.2. Let V be a $\Bbbk$-module and consider $V \otimes B$ with structures

$$\left(v \otimes b\right) \cdot b' := v \otimes bb' \qquad \text{and} \qquad \delta\left(v \otimes b\right) := \sum \left(v \otimes b_{(1)}\right) \otimes b_{(2)}$$

for all $v \in V$, $b \in B$. Then $V \otimes B$ is a Hopf module, since

$$\delta\left(v \otimes bb'\right) = \sum \left(v \otimes b_{(1)} b'_{(1)}\right) \otimes b_{(2)} b'_{(2)} = \sum \left(v \otimes b_{(1)}\right) \cdot b'_{(1)} \otimes b_{(2)} b'_{(2)} = \delta\left(v \otimes b\right) \cdot b'$$

for all $v \in V$, $b \in B$.When we would like to stress the particular actions or coactions we are using, we may denote it by $V \otimes B_{\bullet}^{\bullet}$ as well.

Lemma 1.3. *The assignment $- \otimes B : \mathfrak{M} \to \mathfrak{M}_B^B$ acting on objects as $V \mapsto V \otimes B$ and on morphisms as $f \mapsto f \otimes B$ is functorial.*

Proof. Exercise $\square$

Definition 1.4. Given a coalgebra C and a C-comodule N, the $\Bbbk$-module

$$N^{\mathrm{co}C} := \left\{n \in N \mid \delta\left(n\right) = n \otimes 1\right\}$$

is called the *module of coinvariant elements* (or simply the *module of coinvariants*).

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These notes has been written as an add-on to Chapter 3 of [4] for the course on Hopf Algebras and Quantum Groups held at the VUB during Spring 2019. *They are not intended to substitute the main reference* [4]. The main sources for the material contained here are [2, 4]. Every mistake is due exclusively to the author: please, email to paolo.saracco@ulb.ac.be any correction, suggestion or comment.

Proposition 1.5. *The assignment* $(-)^{\mathsf{co}B} : \mathfrak{M}_B^B \to \mathfrak{M}$ *acting on object via* $M \mapsto M^{\mathsf{co}B}$ *and on morphisms by sending* $f : M \to N$ *to its (co)restriction* $f^{\mathsf{co}B} : M^{\mathsf{co}B} \to N^{\mathsf{co}B}$ *is functorial and it is right adjoint to the functor* $- \otimes B$ *of Lemma 1.3. Unit and counit are given by*

$$\eta_V : V \to (V \otimes B)^{\mathsf{co}B}, \qquad v \mapsto v \otimes 1,$$
$$\epsilon_M : M^{\mathsf{co}B} \otimes B \to M, \qquad \sum_i m_i \otimes b_i \mapsto \sum_i m_i \cdot b_i.$$

Moreover, η *is always a natural isomorphism and hence* $- \otimes B$ *is full and faithful.*

Proof. First of all, let $f : M \to N$ be a morphism of Hopf modules and consider $m \in M^{\mathsf{co}B}$. Then

$$\delta_N(f(m)) = \sum f(m)_{[0]} \otimes f(m)_{[1]} = \sum f\left(m_{[0]}\right) \otimes m_{[1]} = f(m) \otimes 1,$$

so that $f(m) \in N^{\mathsf{co}B}$. As a consequence, f induces $f^{\mathsf{co}B} : M^{\mathsf{co}B} \to N^{\mathsf{co}B}$ as in the statement. It is not difficult to check that the given assignments provide a well-defined functor. To show that it is right adjoint to $- \otimes B$, let us check the two triangular identities:

$$\epsilon_M^{\mathsf{co}B}(\eta_{M^{\mathsf{co}B}}(m)) = \epsilon_M^{\mathsf{co}B}(m \otimes 1) = m,$$
$$\epsilon_{V \otimes B}\left((\eta_V \otimes B)\left(\sum v_i \otimes b_i\right)\right) = \epsilon_{V \otimes B}\left(\sum (v_i \otimes 1) \otimes b_i\right) = \sum v_i \otimes b_i,$$

whence both compositions give an identity morphism and the verifications are complete. Concerning the last claim, notice that for every $\sum v_i \otimes b_i \in (V \otimes B)^{\mathsf{co}B}$ we have

$$\sum v_i \otimes b_{i_{(1)}} \otimes b_{i_{(2)}} = \sum v_i \otimes b_i \otimes 1.$$

By applying $V \otimes \varepsilon \otimes B$ to both sides of the latter relation, we get

$$\sum v_i \otimes b_i = \sum v_i \varepsilon(b_i) \otimes 1.$$

Therefore, the following computations,

$$(V \otimes \varepsilon)(\eta_V(v)) = (V \otimes \varepsilon)(v \otimes 1) = v \qquad \text{and}$$
$$\eta_V\left((V \otimes \varepsilon)\left(\sum v_i \otimes b_i\right)\right) = \eta_V\left(\sum v_i \varepsilon(b_i)\right) = \sum v_i \varepsilon(b_i) \otimes 1 = \sum v_i \otimes b_i,$$

show that $V \otimes \varepsilon$ is the two-sided inverse of η_V for every $V \in \mathfrak{M}$. The additional fact that $- \otimes B$ is full and faithful is a direct consequence of the invertibility of η (see [7] for additional details on this property). □

In general, however, ϵ_M is not an isomorphism, as the following example shows.

Example 1.6. Consider $B := \Bbbk[X]$, the polynomial bialgebra with $\Delta(X) = X \otimes X$ and $\varepsilon(X) = 1$. A module over B is simply a $\Bbbk$-module together with a distinguished endomorphism ξ (representing the acton of X).

We claim that a B-comodule is the same as an $\mathbb{N}$-graded $\Bbbk$-module. Let N be a B-comodule and consider the following $\Bbbk$-submodules $N_p := \{n \in N \mid \delta(n) = n \otimes X^p\}$ for all $p \in \mathbb{N}$. Obviously, $\sum_{p \in \mathbb{N}} N_p \subseteq N$. However, observe the following. For every $n \in N$ we have that $\delta(n) = \sum_{p\in\mathbb{N}} n_p \otimes X^p$ because $\Bbbk[X]$ is free over $\Bbbk$ with basis $\{X^p \mid p \in \mathbb{N}\}$. Denote by $\gamma_p : B \to \Bbbk$ their dual maps (i.e. $\gamma_q(X^p) = \delta_{p,q}$). In addition, coassociativity implies that

$$\sum_{p\in\mathbb{N}} \delta(n_p) \otimes X^p = \sum_{p\in\mathbb{N}} n_p \otimes X^p \otimes X^p$$

and, by applying $N \otimes B \otimes \gamma_q$ to both sides of the latter equality, we get that for every $p \in \mathbb{N}$

$$\delta(n_p) = n_p \otimes X^p,$$

so that $N \subseteq \sum_{p\in\mathbb{N}} N_p$ as well and hence $N = \sum_{p\in\mathbb{N}} N_p$. This allows us to consider the surjective $\Bbbk$-linear morphism $\bigoplus_{p\in\mathbb{N}} N_p \to N$ given by $(n_p)_{p\in\mathbb{N}} \mapsto \sum_{p\in\mathbb{N}} n_p$. Its kernel is given those $(n_p)_{p\in\mathbb{N}}$ such that $\sum_{p\in\mathbb{N}} n_p = 0$. However,

$$0 = (N \otimes \gamma_q)\left(\delta\left(\sum_{p\in\mathbb{N}} n_p\right)\right) = (N \otimes \gamma_q)\left(\sum_{p\in\mathbb{N}} n_p \otimes X^p\right) = n_q$$

for every $q \in \mathbb{N}$ tells us that $(n_p)_{p\in\mathbb{N}}$ has to be 0 and hence $\bigoplus_{p\in\mathbb{N}} N_p \cong N$. The converse is true as well. If $\bigoplus_{p\in\mathbb{N}} N_p$ is any $\mathbb{N}$-graded $\Bbbk$-module we can define a B-coaction by setting $\delta(n_p) := n_p \otimes X^p$ for every $n_p \in N_p$.

Now, a Hopf B-module is an $\mathbb{N}$-graded $\Bbbk$-module $\bigoplus_{p\in\mathbb{N}} M_p = M$ (since it is a comodule) together with a distinguished endomorphism $\xi : M \to M$ (since it is a module) that is compatible with the comodule structure in the sense that

$$\delta(\xi(m_p)) = \delta(m_p \cdot X) = \delta(m_p) \cdot X = m_p \cdot X \otimes X^{p+1} = \xi(m_p) \otimes X^{p+1}$$

for every $p \in \mathbb{N}$ and $m_p \in M_p$. In light of the definition of the M_p's, this tells us that ξ has to be homogeneous of degree $+1$. The converse is true as well: any $\mathbb{N}$-graded $\Bbbk$-module together with a homogeneous endomorphism of degree $+1$ admits a natural structure of Hopf B-module. In this context it is easy to see that $M^{\mathrm{co}B} = M_0$.

Consider the distinguished Hopf module $B \otimes B$ with structures

$$\delta(a \otimes b) := \sum \left(a \otimes b_{(1)}\right) \otimes b_{(2)} \qquad \text{and} \qquad (a \otimes b) \cdot b' := \sum ab'_{(1)} \otimes bb'_{(2)}$$

for all $a, b, b' \in B$. We may denote it by $B_\bullet \otimes B^\bullet_\bullet$ or simply by $B \widehat{\otimes} B$. One may check as before that $\left(B\widehat{\otimes}B\right)^{\mathrm{co}B} \cong B$ via the assignments $\sum a_i \otimes b_i \mapsto \sum a_i \varepsilon(b_i)$ and $b \mapsto b \otimes 1$ for $\sum a_i \otimes b_i \in \left(B\widehat{\otimes}B\right)^{\mathrm{co}B}$ and $b \in B$ respectively. Therefore,

$$\epsilon_{B\widehat{\otimes}B} : B \otimes B^\bullet_\bullet \to B_\bullet \otimes B^\bullet_\bullet : X^a \otimes X^b \mapsto (X^a \otimes 1) \cdot X^b = X^{a+b} \otimes X^b$$

which cannot be surjective, as $1 \otimes X$ cannot be in the image for example.

Theorem 1.7. *(The Structure Theorem of Hopf Modules) The following are equivalent for a bialgebra B:*

(a) *B is a Hopf algebra;*
(b) *$- \otimes B$ and $(-)^{\mathrm{co}B}$ form an (adjoint) equivalence;*
(c) *the $\Bbbk$-linear map $\beta : B \otimes B^\bullet_\bullet \to B_\bullet \otimes B^\bullet_\bullet : a \otimes b \mapsto \sum ab_{(1)} \otimes b_{(2)}$ is an isomorphism of Hopf modules.*

Moreover, if any one of the above holds, $M^{\mathsf{co}B}$ is a $\Bbbk$-direct summand of M for every Hopf module M.

Proof. To prove that (a) implies (b) proceed as follows. For every M define $\tau_M : M \to M : m \mapsto \sum m_{[0]} \cdot S\left(m_{[1]}\right)$ and notice that $\tau_M(m) \in M^{\mathsf{co}B}$ for every $m \in M$. Indeed,

$$\begin{aligned}
\delta\left(\sum m_{[0]} \cdot S\left(m_{[1]}\right)\right) &= \sum m_{[0]_{[0]}} \cdot S\left(m_{[1]}\right)_{(1)} \otimes m_{[0]_{[1]}} S\left(m_{[1]}\right)_{(2)} \\
&= \sum m_{[0]} \cdot S\left(m_{[2]}\right)_{(1)} \otimes m_{[1]} S\left(m_{[2]}\right)_{(2)} \\
&= \sum m_{[0]} \cdot S\left(m_{[2]_{(2)}}\right) \otimes m_{[1]} S\left(m_{[2]_{(1)}}\right) \\
&= \sum m_{[0]} \cdot S\left(m_{[3]}\right) \otimes m_{[1]} S\left(m_{[2]}\right) \\
&= \sum m_{[0]} \cdot S\left(m_{[2]}\right) \otimes m_{[1]_{(1)}} S\left(m_{[1]_{(2)}}\right) \\
&= \sum m_{[0]} \cdot S\left(m_{[1]}\right) \otimes 1.
\end{aligned}$$

Therefore, one may consider the composition

$$\sigma_M := \left(m \overset{\delta}{\mapsto} \sum m_{[0]} \otimes m_{[1]} \overset{\tau_M \otimes B}{\mapsto} \sum m_{[0]} \cdot S\left(m_{[1]}\right) \otimes m_{[2]}\right)$$

and this satisfies

$$\begin{aligned}
\sigma_M\left(\epsilon_M\left(\sum_i m_i \otimes b_i\right)\right) = \sigma_M\left(\sum_i m_i \cdot b_i\right) &= \sum_i \left(m_i \cdot b_i\right)_{[0]} \cdot S\left(\left(m_i \cdot b_i\right)_{[1]}\right) \otimes \left(m_i \cdot b_i\right)_{[2]} \\
&= \sum_i m_i \cdot b_{i_{(1)}} \cdot S\left(b_{i_{(2)}}\right) \otimes b_{i_{(3)}} = \sum_i m_i \otimes b_i
\end{aligned}$$

because the m_i's are coinvariants and

$$\epsilon_M\left(\sigma_M(m)\right) = \epsilon_M\left(\sum m_{[0]} \cdot S\left(m_{[1]}\right) \otimes m_{[2]}\right) = \sum m_{[0]} \cdot S\left(m_{[1]}\right) m_{[2]} = m,$$

whence σ_M is the inverse of ϵ_M.

To prove that (b) implies (c) it suffices to note that, as we already saw implicitly in the foregoing example, β is the composition

$$\beta : \left(B \otimes B_\bullet^\bullet \cong \left(B_\bullet \otimes B_\bullet^\bullet\right)^{\mathsf{co}B} \otimes B_\bullet^\bullet \overset{\epsilon_{B \widehat{\otimes} B}}{\to} B_\bullet \otimes B_\bullet^\bullet\right).$$

Finally, to prove that (c) implies (a) assume that β is invertible and consider $s(a) := (B \otimes \varepsilon)\left(\beta^{-1}(1 \otimes a)\right)$ for all $a \in B$. Since β is colinear, β^{-1} is colinear as well, so that

$$\sum \beta^{-1}\left(1 \otimes a_{(1)}\right) \otimes a_{(2)} = (B \otimes \Delta)\left(\beta^{-1}(1 \otimes a)\right)$$

and, by applying $B \otimes \varepsilon \otimes B$ to both sides,

$$\sum s\left(a_{(1)}\right) \otimes a_{(2)} = \beta^{-1}(1 \otimes a)$$

for every $a \in B$. By applying $(B \otimes \varepsilon) \circ \beta$ to both sides

$$\sum s\left(a_{(1)}\right) a_{(2)} = (B \otimes \varepsilon)\left(\sum s\left(a_{(1)}\right) a_{(2)} \otimes a_{(3)}\right) = (B \otimes \varepsilon)\left(\beta\left(\sum s\left(a_{(1)}\right) \otimes a_{(2)}\right)\right)$$

$$= (B \otimes \varepsilon)\left(\beta\left(\beta^{-1}(1 \otimes a)\right)\right) = (B \otimes \varepsilon)(1 \otimes a) = \varepsilon(a)\,1.$$

For the other relation, observe that $\beta(a'a \otimes b) = a'ab_{(1)} \otimes b_{(2)} = a'\beta(a \otimes b)$ for every $a, a', b \in B$, whence β is also B-linear with respect to the regular left B-action. This allows us to check directly that

$$\begin{aligned}
\sum a_{(1)} s\left(a_{(2)}\right) &= \sum a_{(1)} (B \otimes \varepsilon)\left(\beta^{-1}\left(1 \otimes a_{(2)}\right)\right) = \sum (B \otimes \varepsilon)\left(a_{(1)} \cdot \beta^{-1}\left(1 \otimes a_{(2)}\right)\right) \\
&= \sum (B \otimes \varepsilon)\left(\beta^{-1}\left(a_{(1)} \cdot \left(1 \otimes a_{(2)}\right)\right)\right) = \sum (B \otimes \varepsilon)\left(\beta^{-1}\left(a_{(1)} \otimes a_{(2)}\right)\right) \\
&= \sum (B \otimes \varepsilon)\left(\beta^{-1}((1 \otimes 1) \cdot a)\right) = \sum (B \otimes \varepsilon)\left(\beta^{-1}(1 \otimes 1) \cdot a\right) \\
&= \sum (B \otimes \varepsilon)\left(\beta^{-1}(1 \otimes 1)\right)\varepsilon(a) = \varepsilon(a)\,1
\end{aligned}$$

for every $a \in B$. We are left to check the last claim of the statement. Consider again the morphism $\tau_M : M \to M^{\mathsf{co}B} : m \mapsto \sum m_{[0]} \cdot S\left(m_{[1]}\right)$ from above. This turns out to be a $\Bbbk$-linear retraction of the canonical inclusion $M^{\mathsf{co}B} \to M$, which means that the short exact sequence of $\Bbbk$-modules $0 \to M^{\mathsf{co}B} \to M \to M/M^{\mathsf{co}B} \to 0$ splits and hence $M \cong M^{\mathsf{co}B} \oplus M/M^{\mathsf{co}B}$. The proof is now complete.

$\square$

Example 1.8. Let G be a group and consider $B = \Bbbk G$, the group algebra over G. Analogously to what happens for $\Bbbk[X]$, comodules over B are G-graded $\Bbbk$-modules, that is to say, M is a B-comodule if and only if $M = \bigoplus_{g \in G} M_g$ for some submodules $M_g \subseteq M$. Indeed, assume that M is a B-comodule and define $M_g := \{m \in M \mid \delta(m) = m \otimes e_g\}$. For every $m \in M$ we can write

$$\delta(m) = \sum_{g \in G} m_g \otimes e_g$$

so that $m = \sum_{g \in G} m_g$. Now, by coassociativity of δ we have that

$$\sum_{g \in G} \delta(m_g) \otimes e_g = \sum_{g \in G} m_g \otimes \Delta(e_g) = \sum_{g \in G} m_g \otimes e_g \otimes e_g.$$

Consider the elements $e_g^* : \Bbbk G \to \Bbbk : e_h \longmapsto \delta_{g,h}$ and apply $M \otimes B \otimes e_h^*$ to both sides of the latter equality to see that $\delta(m_h) = m_h \otimes e_h$, i.e. $m_h \in M_h$ and $M = \sum_{g \in G} M_g$. In the same way as we did previously, one proves that the sum is direct, whence $M = \bigoplus_{g \in G} M_g$. Conversely, if $M = \bigoplus_{g \in G} M_g$ then it is enough to define $\delta(m_g) := m_g \otimes e_g$ for every $m_g \in M_g$. Furthermore, M is a Hopf module if and only if the coaction is B-linear, which can be seen to be equivalent to the fact that $M_g \cdot e_h \subseteq M_{gh}$ for every $g, h \in G$. Now,

$$M_g \cdot e_h \subseteq M_{gh} = M_{gh} \cdot (e_{h^{-1}} e_h) \subseteq M_{ghh^{-1}} \cdot e_h = M_g \cdot e_h$$

entails that in fact $M_g \cdot e_h = M_{gh}$ for every $g, h \in G$ and, in particular, that $M_g = M_1 \cdot e_g$ for every $g \in G$. Therefore, for every $m \in M$ there exist $\mu_g \in M_1$ for $g \in G$ such that $m = \sum_{g \in G} \mu_g \cdot e_g$. Let us see that these are also unique. Assume that

$$\sum_{g \in G} \mu'_g \cdot e_g = m = \sum_{g \in G} \mu_g \cdot e_g$$

and apply δ to both sides to get

$$\sum_{g\in G}\mu'_g\cdot e_g\otimes e_g=\sum_{g\in G}\mu_g\cdot e_g\otimes e_g.$$

Now apply $M\otimes e_h^*$ to get that $\mu'_h\cdot e_h=\mu_h\cdot e_h$ and finally act with $e_{h^{-1}}$ to conclude that $\mu'_h=\mu_h$ for every $h\in G$.

Observe now that $M^{\mathsf{co}B}=M_1$ again. In fact, $\delta(m)=\sum_{g\in G}\delta(m_g)=\sum_{g\in G}m_g\otimes e_g=m\otimes e_1$ if and only if $m=M_1$, because the e_g's are $\Bbbk$-linearly independent. Therefore the morphism

$$\epsilon_M:M_1\otimes B\to M:\sum_i m_i\otimes e_{g_i}\longmapsto\sum_i m_i\cdot e_{g_i}$$

is invertible with inverse $M\to M_1\otimes B:m=\sum_{g\in G}\mu_g\cdot e_g\longmapsto\sum_{g\in G}\mu_g\otimes e_g$. This is coherent with the Structure Theorem since for every $m=\sum_{g\in G}\mu_g\cdot e_g\in M$ we have

$$\sum_{g\in G}(\mu_g\cdot e_g)_{[0]}S\left((\mu_g\cdot e_g)_{[1]}\right)\otimes(\mu_g\cdot e_g)_{[2]}=\sum_{g\in G}\mu_g\cdot e_gS(e_g)\otimes e_g=\sum_{g\in G}\mu_g\otimes e_g.$$

2. Rational modules

Recall that, in general, if C is a $\Bbbk$-coalgebra then C^* is a $\Bbbk$-algebra with unit ε and multiplication the convolution product $f*g=(f\otimes g)\circ\Delta$. From now on and until the end of the section, coalgebras are additionally assumed to be *free as a $\Bbbk$-modules* (unless stated otherwise).

Lemma 2.1. *For every $\Bbbk$-module M and every $\Bbbk$-coalgebra C, every linear map $\delta:M\to M\otimes C$ induces a linear map $\mu_\delta:C^*\otimes M\to M$ given by*

$$\mu_\delta(f\otimes m)=(M\otimes f)(\delta(m)) \tag{1}$$

for all $f\in C^*$, $m\in M$. Moreover, δ is a coassociative and counital coaction if and only if μ_δ is an associative and unital action.

Proof. The map μ_δ is given as the following composition

$$\mu_\delta=(M\otimes\mathsf{ev}_C)\circ(\tau_{C^*,M}\otimes C)\circ(C^*\otimes\delta)$$

where $\tau_{C^*,M}:C^*\otimes M\to M\otimes C^*$ is the twist and $\mathsf{ev}_C:C^*\otimes C\to\Bbbk$ is the evaluation map. Now recall that, by the hom-tensor adjunction, we have a bijection $\rho:\mathsf{Hom}_\Bbbk(C^*\otimes M,M)\to\mathsf{Hom}_\Bbbk(C^*,\mathsf{End}_\Bbbk(M))$ and recall also that $\mu\in\mathsf{Hom}_\Bbbk(C^*\otimes M,M)$ is an associative and unital action if and only if $\rho(\mu)$ is a morphism of $\Bbbk$-algebras. In light of this, for every $f,g\in C^*$ consider the following diagrams,

$$\begin{tikzcd} M \arrow[d, "\delta"'] \arrow[dr, "id_M"] & \\ M\otimes C \arrow[r, "M\otimes\varepsilon"'] & M \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} M \arrow[r, "\delta"] \arrow[d, "\delta"'] & M\otimes C \arrow[r, "M\otimes g"] \arrow[d, "\delta\otimes M"'] & M \arrow[d, "\delta"] \\ M\otimes C \arrow[r, "M\otimes\Delta"'] \arrow[drr, bend right, "M\otimes f*g"'] & M\otimes C\otimes C \arrow[dr, "M\otimes f\otimes g"'] & M\otimes C \arrow[d, "M\otimes f"] \\ & & M \end{tikzcd}$$

where in the latter one everything is commutative apart from the internal square (expressing coassociativity of δ) and the external triangle (expressing multiplicativity of $\rho(\mu_\delta)$, which is equivalent to associativity of μ_δ). The left-most diagram makes it clear that δ is counital if and only if $\rho(\mu_\delta)$ is unital, if and only if μ_δ is unital. It is also easy to see from the right-most one that if δ is coassociative, then μ_δ is associative. To prove the converse, assume that for every $f, g \in C^*$ we have $\rho(\mu_\delta)(f * g) = \rho(\mu_\delta)(f) \circ \rho(\mu_\delta)(g)$. This implies that for every $m \in M$,

$$(M \otimes f \otimes g)((\delta \otimes M)(\delta(m)) - (M \otimes \Delta)(\delta(m))) = 0. \tag{2}$$

Write $(\delta \otimes M)(\delta(m)) - (M \otimes \Delta)(\delta(m)) = \sum_{i,j=1}^{t} m_{ij} \otimes c_i \otimes d_j$ where the c_i's and the d_j's are elements of the $\Bbbk$-basis of C. One can consider $f = c_i^*$ and $g = d_j^*$ for i and j running from 1 to t defined by setting $c_i^*(c_k) = \delta_{i,k} = d_i^*(d_k)$. Thus (2) entails that $m_{ij} = 0$ for all $i, j = 1, \ldots, t$. ☐

This provides for us a functor $\mathcal{L} : \mathfrak{M}^C \to {}_{C^*}\mathfrak{M}$ from the category of right C-comodules $\mathfrak{M}^C$ to the one of left C^*-modules ${}_{C^*}\mathfrak{M}$. In general this is not an equivalence, as we will see soon. In this section we are interested in describing those C^*-modules that lies in the image of $\mathcal{L}$. To this aim, we start by giving the following definition.

Definition 2.2. A C^*-module (M, μ) is said to be *rational* if there exists a linear map $\delta : M \to M \otimes C$, called the *associated coaction*, such that $\mu = \mu_\delta$.

Remark 2.3. In light of Lemma 2.1, the associated coaction of a rational module (M, μ) is indeed a coaction, whence $(M, \mu) = \mathcal{L}(M, \delta)$.

Example 2.4. Since (C, Δ) is a C-comodule, C is a rational C^*-module with action $f \cdot c = \sum c_{(1)} f\left(c_{(2)}\right)$. Analogously, for every $\Bbbk$-module M we have that $(M \otimes C, M \otimes \Delta)$ is a C-comodule and hence a rational C^*-module with action $f \cdot (m \otimes c) = \sum m \otimes c_{(1)} f\left(c_{(2)}\right)$.

Our motivation for introducing rational modules will be clear in the section devoted to integrals for Hopf algebras. For every $\Bbbk$-module M and every $\Bbbk$-coalgebra C consider the following map

$$\alpha_M : M \otimes C \to \mathsf{Hom}_\Bbbk(C^*, M) : m \otimes c \mapsto [f \mapsto m f(c)],$$

that is to say,

$$\alpha_M(m \otimes c)(f) = (M \otimes f)(m \otimes c) \tag{3}$$

Lemma 2.5. *The* $\Bbbk$*-linear morphism* α_M *is injective.*

Proof. Notice that α_M is indeed a $\Bbbk$-linear morphism. In fact, it can be seen as the composition

$$\begin{gathered} M \otimes C \to M \otimes C^{**} \to \mathsf{Hom}_\Bbbk(C^*, M) \\ m \otimes c \mapsto m \otimes \mathsf{ev}_c \mapsto [f \mapsto \mathsf{ev}_c(f)\, m] \end{gathered}$$

Moreover, let $\sum_{i=1}^{t} m_i \otimes c_i \in M \otimes C$ be such that $\sum m_i f(c_i) = 0$ for all $f \in C^*$, where again we may assume that the c_i's are elements of the $\Bbbk$-basis of C. Analogously to what we did before, we may consider $f = c_i^*$ for $i = 1, \ldots, t$ and this entails that $m_i = 0$ for every i. $\square$

Corollary 2.6. *The assignment* $\gamma_M : \mathsf{Hom}_{\Bbbk}(M, M \otimes C) \to \mathsf{Hom}_{\Bbbk}(C^* \otimes M, M)$ *provided by* (1) *is injective. In particular if the associated coaction exists, then it is unique.*

Proof. Notice that for every $m \in M$, $f \in C^*$, $\delta \in \mathsf{Hom}_{\Bbbk}(M, M \otimes C)$

$$((\alpha_M \circ \delta)(m))(f) \stackrel{(3)}{=} (M \otimes f)(\delta(m)) \stackrel{(1)}{=} \mu_\delta(f \otimes m) = \gamma_M(\delta)(f \otimes m),$$

whence if $\gamma_M(\delta) = 0$ then $\alpha_M \circ \delta = 0$ and so, by injectivity of α_M, $\delta = 0$. $\square$

On the other hand, for every C^*-module M we may consider the assignment

$$\beta_M : M \to \mathsf{Hom}_{\Bbbk}(C^*, M) : m \mapsto [f \mapsto f \cdot m],$$

where $\cdot$ denotes the C^* action.

Remark 2.7. Note that $\mathsf{Hom}_{\Bbbk}(C^*, M)$ is a C^*-module with action

$$C^* \otimes \mathsf{Hom}_{\Bbbk}(C^*, M) \to \mathsf{Hom}_{\Bbbk}(C^*, M) : f \otimes \psi \mapsto [(f \cdot \psi) : g \mapsto \psi(g * f)].$$

Lemma 2.8. *Both* α_M *and* β_M *are morphisms of* C^**-modules. Moreover, they are natural transformations.*

Proof. The following computations for $f, g \in C^*, m \in M, c \in C$

$$\begin{aligned}\alpha_M(f \cdot (m \otimes c))(g) &= \sum \alpha_M\left(m \otimes c_{(1)} f\left(c_{(2)}\right)\right)(g) = \sum m g\left(c_{(1)}\right) f\left(c_{(2)}\right) \\ &= \sum m (g * f)(c) = \alpha_M(m \otimes c)(g * f) = (f \cdot \alpha_M(m \otimes c))(g), \\ \beta_M(f \cdot m)(g) &= g \cdot (f \cdot m) = (g * f) \cdot m = \beta_M(m)(g * f) = (f \cdot \beta_M(m))(g),\end{aligned}$$

entail that both α_M and β_M are C^*-linear. Concerning naturality, if $\psi : M \to N$ is a C^*-linear morphism then

$$\begin{aligned}((\alpha_N \circ (\psi \otimes C))(m \otimes c))(f) &= \alpha_N(\psi(m) \otimes c)(f) \stackrel{(3)}{=} (M \otimes f)(\psi(m) \otimes c) = \psi(m) f(c) \\ &= \psi(m f(c)) = \psi(\alpha_M(m \otimes c)(f)) \\ &= ((\mathsf{Hom}_{\Bbbk}(C^*, \psi) \circ \alpha_M)(m \otimes c))(f), \\ ((\beta_N \circ \psi)(m))(f) &= f \cdot \psi(m) = \psi(f \cdot m) = \psi(\beta_M(m)(f)) \\ &= ((\mathsf{Hom}_{\Bbbk}(C^*, \psi) \circ \beta_M)(m))(f)\end{aligned}$$

for every $m \in M, c \in C, f \in C^*$. $\square$

Proposition 2.9. *The following are equivalent for a* C^**-module* M*:*

(a) *there exists* $\delta : M \to M \otimes C$ *such that* $\alpha_M \circ \delta = \beta_M$*;*
(b) *there exists* $\delta : M \to M \otimes C$ *such that* $\mu_M = \mu_\delta$ *(i.e.* M *is a rational* C^**-module).*

Proof. Notice that for every $m \in M$, $f \in C^*$,

$$((\alpha_M \circ \delta)(m))(f) \stackrel{(3)}{=} (M \otimes f)(\delta(m)) \stackrel{(1)}{=} \mu_\delta(f \otimes m)$$

and

$$(\beta_M(m))(f) = \mu_M(f \otimes m).$$

Thus $\alpha_M \circ \delta = \beta_M$ if and only if for every $m \in M$, $f \in C^*$ we have $\mu_\delta(f \otimes m) = \mu_M(f \otimes m)$, if and only if $\mu_\delta = \mu_M$. Now, in light of Lemma 2.12 we know that M^{rat} satisfies condition (1) with $\delta : M^{\mathsf{rat}} \to M^{\mathsf{rat}} \otimes C : m \mapsto \sum_{i=1}^t m_i \otimes c_i$, which implies that $\mu_{M^{\mathsf{rat}}} = \mu_\delta$ by condition (2) and hence that δ is a coassociative and counital coaction by Lemma 2.1. $\square$

Definition 2.10. For every C^*-module M we define $M^{\mathsf{rat}} := \beta_M^{-1}(\alpha_M(M \otimes C))$ and we call it the *rational part* of M.

In what follows we are going to show that M^{rat} is always a rational C^*-module and that it is the maximal rational C^*-module in M (i.e. the biggest one whose induced C^*-action is coming from a C-coation as in Lemma 2.1).

Lemma 2.11. *For every C^*-module M, M^{rat} is a C^*-submodule of M. In particular, it is a C^*-module.*

Proof. Notice that on the one hand $M \otimes C$ with $\delta = M \otimes \Delta$ is a C-comodule, whence it becomes a C^*-module with action μ_δ as in Lemma 2.1. On the other hand, $\mathsf{Hom}_{\Bbbk}(C^*, M)$ is a C^*-module via

$$C^* \otimes \mathsf{Hom}_{\Bbbk}(C^*, M) \to \mathsf{Hom}_{\Bbbk}(C^*, M) : f \otimes \psi \longmapsto [g \longmapsto \psi(g * f)].$$

Now, the following computations for $f, g \in C^*, m \in M, c \in C$

$$\begin{aligned}
\alpha_M(f \cdot (m \otimes c))(g) &= \sum \alpha_M\left(m \otimes c_{(1)} f\left(c_{(2)}\right)\right)(g) = \sum mg\left(c_{(1)}\right) f\left(c_{(2)}\right) \\
&= \sum m(g * f)(c) = \alpha_M(m \otimes c)(g * f) = (f \cdot \alpha_M(m \otimes c))(g), \\
\beta_M(f \cdot m)(g) &= \sum \beta_M\left(m_{[0]} f\left(m_{[1]}\right)\right)(g) = \sum \left(g \cdot m_{[0]}\right) f\left(m_{[1]}\right) \\
&= \sum m_{[0]} g\left(m_{[1]}\right) f\left(m_{[2]}\right) = \sum m_{[0]}(g * f)\left(m_{[1]}\right) \\
&= \beta_M(m)(g * f) = (f \cdot \beta_M(m))(g),
\end{aligned}$$

entails that both α_M and β_M are C^*-linear. Thus M^{rat} is a C^*-submodule of M. $\square$

Lemma 2.12. *For every C^*-module M and $m \in M$, $m \in M^{\mathsf{rat}}$ if and only if there exists a (necessarily unique) $\sum_{i=1}^t m_i \otimes c_i$ in $M \otimes C$ such that $f \cdot m = \sum_{i=1}^t m_i f(c_i)$ for every $f \in C^*$. Moreover, $\sum_{i=1}^t m_i \otimes c_i$ lives in $M^{\mathsf{rat}} \otimes C$. In particular, M^{rat} is a rational C^*-module and a C-comodule.*

Proof. Observe that $f \cdot m = (\beta_M(m))(f)$ and $\sum_{i=1}^t m_i f(c_i) = \left(\alpha_M\left(\sum_{i=1}^t m_i \otimes c_i\right)\right)(f)$, thus we have that $m \in M^{\mathsf{rat}}$ if and only if $\beta_M(m) \in \alpha_M(M \otimes C)$, if and only if there exists $\sum_{i=1}^t m_i \otimes c_i \in M \otimes C$ such that $\beta_M(m) = \alpha_M\left(\sum_{i=1}^t m_i \otimes c_i\right)$, if and only if $f \cdot m = \sum_{i=1}^t m_i f(c_i)$ for every $f \in C^*$. Uniqueness follows immediately from injectivity of α_M. Concerning the fact that $\sum_{i=1}^t m_i \otimes c_i \in M^{\mathsf{rat}} \otimes C$, note firstly that since C is free over $\Bbbk$, it is flat and hence we may consider $M^{\mathsf{rat}} \otimes C \subseteq M \otimes C$. Secondly, since M^{rat} is a C^*-submodule of M, for every $m \in M^{\mathsf{rat}}$ and $f \in C^*$ we have $\sum_{i=1}^t m_i f(c_i) = f \cdot m \in M^{\mathsf{rat}}$.

By assuming the c_i's to be part of a $\Bbbk$-basis for C and by taking $f = c_i^*$ for i from 1 to t we can conclude that $m_i \in M^{\mathsf{rat}}$ for all i. $\square$

Corollary 2.13. *M^{rat} is the maximal rational C^*-module in M. In fact, it is the sum of all rational C^*-modules in M.*

Proof. Let N be a C^*-submodule of M which is rational and denote by $j : N \to M$ the canonical inclusion. Then there exists $\delta : N \to N \otimes C$ such that $\alpha_N \circ \delta = \beta_N$. Set $j_* := \mathsf{Hom}_{\Bbbk}(C^*, j)$. By naturality of α and β we get

$$\alpha_M \circ (j \otimes C) \circ \delta = j_* \circ \alpha_N \circ \delta = j_* \circ \beta_N = \beta_M \circ j,$$

from which it follows that $N \subseteq M^{\mathsf{rat}}$ and the first claim is proved. As a consequence, every rational C^*-module in M lies in M^{rat} and so does their sum. However, being M^{rat} rational, it is one of the modules appearing in the sum, thus proving the second claim. $\square$

Remark 2.14. For the interested reader, M^{rat} can be seen as the pullback of the maps (α_M, β_M). Corollary 2.13 becomes then a consequence of this fact.

We are now ready to see why the functor $\mathcal{L}$ is not an equivalence in general. For every C^*-module (M, μ), consider its rational part M^{rat} together with the coaction $\delta_\mu : M^{\mathsf{rat}} \to M^{\mathsf{rat}} \otimes C$ of Lemma 2.12. Now, let $\varphi : (M, \mu) \to (N, \nu)$ be a morphism of C^*-modules and denote by φ_* the C^*-linear morphism $\mathsf{Hom}_{\Bbbk}(C^*, \varphi) : \mathsf{Hom}_{\Bbbk}(C^*, M) \to \mathsf{Hom}_{\Bbbk}(C^*, N), \psi \mapsto \varphi \circ \psi$.

Lemma 2.15. *The C^*-linear morphism φ induces a C-colinear morphism $\varphi^{\mathsf{rat}} : (M^{\mathsf{rat}}, \delta_\mu) \to (N^{\mathsf{rat}}, \delta_\nu)$.*

Proof. Let $m \in M^{\mathsf{rat}}$ and consider $\varphi(m) \in N$. Since α and β are natural transformations,

$$\beta_N(\varphi(m)) = \varphi_*(\beta_M(m)) = \varphi_*(\alpha_M(\delta_\mu(m))) = \alpha_N((\varphi \otimes C)(\delta_\mu(m))) \tag{4}$$

whence $\varphi(m) \in N^{\mathsf{rat}}$. Moreover, from this it follows that $\beta_N(\varphi(m)) = \alpha_N(\delta_\nu(\varphi(m)))$ and hence, by injectivity of α_N, that $(\varphi \otimes C)(\delta_\mu(m)) = \delta_\nu(\varphi(m))$. Denote by $\varphi^{\mathsf{rat}} : M^{\mathsf{rat}} \to N^{\mathsf{rat}}$ the (co)restriction of φ to M^{rat} and N^{rat}. Since $m \in M^{\mathsf{rat}}$ was general, we conclude that $(\varphi^{\mathsf{rat}} \otimes C) \circ \delta_\mu = \delta_\nu \circ \varphi^{\mathsf{rat}}$. $\square$

Proposition 2.16. *The assigment $\mathcal{R} : {}_{C^*}\mathfrak{M} \to \mathfrak{M}^C, (M, \mu) \mapsto (M^{\mathsf{rat}}, \delta_\mu)$ is functorial and it is right adjoint to the functor $\mathcal{L} : \mathfrak{M}^C \to {}_{C^*}\mathfrak{M} : (N, \delta) \mapsto (N, \mu_\delta)$. The unit is given by the identity morphism and the counit by the canonical inclusion $M^{\mathsf{rat}} \subseteq M$.*

Proof. Functoriality follows from Lemma 2.15. To prove that it is right adjoint to $\mathcal{L}$, let us start by observing that if (N, δ) is a C-comodule then (N, μ_δ) is a rational C^*-module (by definition) and δ is the unique associated coaction to μ_δ. Thus $\mathcal{R}(N, \mu_\delta) = (N, \delta)$. The other way around, let (M, μ) be any C^*-module. Then $\mathcal{R}(M, \mu) = (M^{\mathsf{rat}}, \delta_\mu)$ where δ_μ is the (unique) coaction such that $\alpha_{M^{\mathsf{rat}}} \circ \delta_\mu = \beta_{M^{\mathsf{rat}}}$. Therefore, $\mathcal{L}\mathcal{R}(M, \mu) = \mathcal{L}(M^{\mathsf{rat}}, \delta_\mu) = \left(M^{\mathsf{rat}}, \mu_{\delta_\mu}\right)$ where $\mu_{\delta_\mu} = \gamma_{M^{\mathsf{rat}}}(\delta_\mu)$ satisfies

$$\mu_{\delta_\mu}(f \otimes m) \overset{(1)}{=} (M^{\mathsf{rat}} \otimes f)(\delta_\mu(m)) \overset{(3)}{=} \alpha_{M^{\mathsf{rat}}}(\delta_\mu(m))(f) = \beta_{M^{\mathsf{rat}}}(m)(f) = \mu_{M^{\mathsf{rat}}}(f \otimes m)$$

for every $m \in M^{\mathsf{rat}}$ and $f \in C^*$. Therefore, $\mathcal{LR}(M,\mu) = (M^{\mathsf{rat}}, \mu_{M^{\mathsf{rat}}})$. Let $\epsilon_M : (M^{\mathsf{rat}}, \mu_{M^{\mathsf{rat}}}) \to (M,\mu)$ denote the canonical inclusion. Then $\eta_N = \mathsf{Id} : (N,\delta) \to \mathcal{RL}(N,\delta)$ and ϵ_M are the unit and the counit of the stated adjunction, respectively. To show this, observe simply that $\mathcal{R}(\epsilon_M) = \epsilon_M^{\mathsf{rat}} = \mathsf{Id}_{(M^{\mathsf{rat}},\delta_\mu)}$ and that $\epsilon_{\mathcal{L}(N)} : (N,\mu_\delta)^{\mathsf{rat}} = (N,\mu_\delta) \to (N,\mu_\delta) = \mathsf{Id}_{(N,\mu_\delta)}$. □

Theorem 2.17. *The functor $\mathcal{L} : \mathfrak{M}^C \to {}_{C^*}\mathfrak{M} : (N,\delta) \mapsto (N,\mu_\delta)$ is an equivalence of categories (in fact, an isomorphism) if and only if the coalgebra C is free of finite rank.*

Proof. In light of Proposition 2.16, $\mathcal{L}$ is an (adjoint) equivalence if and only if the counit is a natural isomorphism, that is to say, if and only if every C^*-module M is rational. Consider the distinguished C^*-module $(C^*, *)$. In light of Proposition 2.9, saying that it is rational means that there should exist $\delta : C^* \to C^* \otimes C$ such that $\alpha_{C^*} \circ \delta = \beta_{C^*}$. Therefore, for every $f \in C^*$ there exists $\sum f_{[0]} \otimes f_{[1]} \in C^* \otimes C$ such that $h * f = \sum f_{[0]} h\left(f_{[1]}\right)$ for all $h \in C^*$. Pick $f = \varepsilon$. Then the latter relation would imply that $h = h * \varepsilon = \sum \varepsilon_{[0]} h\left(\varepsilon_{[1]}\right) = \sum \varepsilon_{[0]} \mathsf{ev}_{\varepsilon_{[1]}}(h)$ for all $h \in C^*$, that is to say, that C^* is a finitely generated and projective $\Bbbk$-module with dual basis $\sum \varepsilon_{[0]} \otimes \mathsf{ev}_{\varepsilon_{[1]}}$. Write $\delta(\varepsilon) = \sum_{i=1}^t g_i \otimes c_i$ in such a way that the c_i's are elements of the $\Bbbk$-basis of C. Consider the canonical (injective, because C is free) $\Bbbk$-linear morphism $j : C \to C^{**} : c \mapsto \mathsf{ev}_c$. Then for every $h \in C^*$ and every $\mathsf{c} \in C$ we have

$$\mathsf{ev}_c(h) = \mathsf{ev}_c\left(\sum_{i=1}^t g_i h(c_i)\right) = \sum_{i=1}^t \mathsf{ev}_c(g_i)\, h(c_i) = \sum_{i=1}^t g_i(c)\, \mathsf{ev}_{c_i}(h),$$

that is to say,

$$j(c) = \mathsf{ev}_c = \sum_{i=1}^t g_i(c)\, \mathsf{ev}_{c_i} = \sum_{i=1}^t g_i(c)\, j(c_i) = j\left(\sum_{i=1}^t g_i(c)\, c_i\right)$$

and hence $c = \sum_{i=1}^t g_i(c)\, c_i$ by injectivity of j. It follows that C is finitely generated and so it is free of finite rank. Conversely, assume that C is free of finite rank. Let $\left\{c_{(1)}, \ldots, c_t\right\}$ be a $\Bbbk$-basis and $\left\{c_{(1)}^*, \ldots, c_t^*\right\}$ be its dual basis. Let M be a C^*-module and for every $m \in M$ set $m_i := c_i^* \cdot m$. Then, for every $f \in C^*$

$$\sum_{i=1}^t m_i f(c_i) = \sum_{i=1}^t (c_i^* \cdot m)\, f(c_i) = \left(\sum_{i=1}^t f(c_i)\, c_i^*\right) \cdot m = f \cdot m$$

and hence $m \in M^{\mathsf{rat}}$ by Lemma 2.12. □

Denote by $\mathfrak{Rat}\left({}_{C^*}\mathfrak{M}\right)$ the full subcategory of rational C^*-modules. We can consider the corestriction $\mathcal{L}' : \mathfrak{M}^C \to \mathfrak{Rat}\left({}_{C^*}\mathfrak{M}\right)$ of the functor $\mathcal{L}$ and the restriction $\mathcal{R}' : \mathfrak{Rat}\left({}_{C^*}\mathfrak{M}\right) \to \mathfrak{M}^C, (M,\mu) \mapsto (M,\delta_\mu)$ of the functor $\mathcal{R}$.

Theorem 2.18. *The functors $\mathcal{L}'$ and $\mathcal{R}'$ are quasi-inverses, giving an equivalence of categories $\mathfrak{M}^C \cong \mathfrak{Rat}\left({}_{C^*}\mathfrak{M}\right)$.*

Remark 2.19. For the interested reader, the foregoing theorem can be used to conclude that the category of comodules over a coalgebra is a Grothendieck category (see [5]).

Remark 2.20. If C is finitely generated and projective as a $\Bbbk$-module with dual basis $\{e_i, e_i^*\}_{i=1}^s$, then we can obtain the same conclusion of Lemma 2.1 by observing that

$$\begin{aligned}\sum_{i,j=1}^{t} m_{ij} \otimes c_i \otimes d_j &= \sum_{i,j=1}^{t} \sum_{k,h=1}^{s} m_{ij} e_k^*(c_i) e_h^*(d_j) \otimes e_k \otimes e_h \\ &= \sum_{k,h=1}^{s} \left(\sum_{i,j=1}^{t} m_{ij} e_k^*(c_i) e_h^*(d_j) \right) \otimes e_k \otimes e_h = 0\end{aligned}$$

implies that $(\delta \otimes M)(\delta(m)) = (M \otimes \Delta)(\delta(m))$. Analogously, Lemma 2.5 can be obtained by observing that

$$\sum_{i=1}^{t} m_i \otimes c_i = \sum_{i=1}^{t} \sum_{k=1}^{s} m_i \otimes e_k^*(c_i) e_k = \sum_{k=1}^{s} \left(\sum_{i=1}^{t} m_i e_k^*(c_i) \right) \otimes e_k = 0.$$

Notice that in this case, α_M is always an isomorphism with inverse

$$\alpha_M^{-1} : \mathsf{Hom}_{\Bbbk}(C^*, M) \to M \otimes C : \psi \mapsto \sum_{k=1}^{s} \psi(e_k^*) \otimes e_k.$$

Indeed,

$$\alpha_M^{-1}(\alpha_M(m \otimes c)) = \sum_{k=1}^{s} \alpha_M(m \otimes c)(e_k^*) \otimes e_k = \sum_{k=1}^{s} m e_k^*(c) \otimes e_k = m \otimes \sum_{k=1}^{s} e_k^*(c) e_k = m \otimes c,$$

$$\alpha_M\left(\alpha_M^{-1}(\psi)\right)(f) = \alpha_M\left(\sum_{k=1}^{s} \psi(e_k^*) \otimes e_k\right)(f) = \sum_{k=1}^{s} \psi(e_k^*) f(e_k) = \psi\left(\sum_{k=1}^{s} f(e_k) e_k^*\right) = \psi(f).$$

For this reason, it makes no sense to speak about rational C^*-modules in this context, as every C^*-module would be rational.

Example 2.21. Assume that $\Bbbk$ is a field and consider $\Bbbk[X]$, the polynomial algebra with the coalgebra (in fact, bialgebra) structure given by

$$\Delta\left(X^t\right) = \sum_{i+j=t} \binom{t}{i} X^i \otimes X^j, \qquad \varepsilon\left(X^t\right) = \delta_{0,t}$$

for all $t \in \mathbb{N}$. For all $s \in \mathbb{N}$ define $\gamma^s : \Bbbk[X] \to \Bbbk : X^i \mapsto \delta_{i,s}$. Notice that

$$\left(\gamma^s * \gamma^t\right)(X^r) = \sum_{i+j=r} \binom{r}{i} \gamma^s\left(X^i\right) \gamma^t\left(X^j\right) = \begin{cases} 0 & s+t \neq r \\ \binom{r}{s} & s+t = r \end{cases},$$

that is to say, $\gamma^s * \gamma^t = \binom{s+t}{s} \gamma^{s+t}$. Moreover, $\gamma^0 = \varepsilon = 1_{\Bbbk[X]^*}$. If M is a rational $\Bbbk[X]^*$-module, then for every $m \in M$ we can write

$$\delta(m) = \sum_{i \geq 0} m_i \otimes X^i$$

with almost all $m_i = 0$. Set $N_m := \max\{i \geq 0 \mid m_i \neq 0\}$, so that $\delta(m) = \sum_{i=0}^{N_m} m_i \otimes X^i$. As a consequence, for every $t \in \mathbb{N}$ we have

$$\gamma^t \cdot m = \sum_{i=0}^{N_m} m_i \gamma^t\left(X^i\right) = \begin{cases} 0 & t > N_m \\ m_t & t \leq N_m \end{cases}.$$

Summing up, if M is a rational $\Bbbk[X]^*$-module, then for every $m \in M$ there exists $N_m \geq 0$ such that $\gamma^{N_m+1} \cdot m = 0$. Observe that this implies that $\gamma^s \cdot m = 0$ for every $s \geq N_m + 1$, since in such a case $\binom{s}{N_m+1}\gamma^s \cdot m = \binom{r+N_m+1}{N_m+1}\gamma^{r+N_m+1} \cdot m = \gamma^r \cdot \left(\gamma^{N_m+1} \cdot m\right) = 0$, from which it follows that $\gamma^s \cdot m = 0$.

Conversely, assume that M is a $\Bbbk[X]^*$-module such that for every $m \in M$ there exists $N_m \geq 0$ (which we may assume to be minimal) satisfying $\gamma^{N_m+1} \cdot m = 0$ and let us show that M is rational. First of all, observe that for all $m, m' \in M$ and $k \in \Bbbk$ we have $N_{km} = N_m$, $N_{m+m'} = \max\{N_m, N_{m'}\}$ and $N_{\gamma^t \cdot m} = N_m - t$, as

$$\gamma^{N_m-t} \cdot \left(\gamma^t \cdot m\right) = \left(\gamma^{N_m-t} * \gamma^t\right) \cdot m = \binom{N_m}{t}\gamma^{N_m} \cdot m = 0$$

and no smaller one satisfies the same property. For every $m \in M$ define

$$\delta(m) := \sum_{i=0}^{N_m} \gamma^i \cdot m \otimes X^i.$$

This δ is $\Bbbk$-linear because

$$\begin{aligned}
\delta(km + m') &= \sum_{i=0}^{\max\{N_m, N_{m'}\}} \gamma^i \cdot (km + m') \otimes X^i \\
&= k \sum_{i=0}^{\max\{N_m, N_{m'}\}} \gamma^i \cdot m \otimes X^i + \sum_{i=0}^{\max\{N_m, N_{m'}\}} \gamma^i \cdot m' \otimes X^i \\
&= k \sum_{i=0}^{N_m} \gamma^i \cdot m \otimes X^i + \sum_{i=0}^{N_{m'}} \gamma^i \cdot m' \otimes X^i \\
&= k\delta(m) + \delta(m')
\end{aligned}$$

and it is counital because

$$(M \otimes \varepsilon)(\delta(m)) = \sum_{i=0}^{N_m} \gamma^i \cdot m\varepsilon\left(X^i\right) = \gamma^0 \cdot m = m,$$

whence we are left to check that it is coassociative. To this aim compute

$$(M \otimes \Delta)(\delta(m)) = \sum_{h=0}^{N_m} \sum_{i+j=h} \binom{h}{i} \gamma^h \cdot m \otimes X^i \otimes X^j$$

and

$$(\delta \otimes \Bbbk[X])(\delta(m)) = \sum_{i=0}^{N_m} \delta\left(\gamma^i \cdot m\right) \otimes X^i = \sum_{i=0}^{N_m} \sum_{j=0}^{N_m-i} \gamma^j \cdot \left(\gamma^i \cdot m\right) \otimes X^j \otimes X^i$$

$$= \sum_{i=0}^{N_m} \sum_{j=0}^{N_m - i} \binom{i+j}{i} \gamma^{i+j} \cdot m \otimes X^j \otimes X^i \overset{(h:=i+j)}{=} \sum_{h=0}^{N_m} \sum_{i+j=h} \binom{h}{i} \gamma^h \cdot m \otimes X^j \otimes X^i.$$

Finally, we want to show that for every $g \in \Bbbk[X]^*$ we have $g \cdot m = \sum_{i=0}^{N_m} (\gamma^i \cdot m)\, g(X^i)$. Observe that the assignment

$$\Bbbk[X]^* \to \Bbbk[[Z]] : f \mapsto \sum \frac{f(X^n)}{n!} Z^n$$

gives an isomorphism of $\Bbbk$-algebras. Note also that, via this isomorphism, γ^n corresponds to $Z^n/n!$ for every $n \geq 0$. As a consequence, we may write $g = \sum_{n \geq 0} g(X^n)\, \gamma^n$. Now, decompose g as follows:

$$\begin{aligned} g &= \sum_{n \geq 0} g(X^n)\, \gamma^n = \sum_{n=0}^{N_m} g(X^n)\, \gamma^n + \sum_{n \geq N_m + 1} g(X^n)\, \gamma^n \\ &= \sum_{n=0}^{N_m} g(X^n)\, \gamma^n + \sum_{t \geq 0} g\left(X^{N_m+1+t}\right) \gamma^{N_m+1+t} \\ &= \sum_{n=0}^{N_m} g(X^n)\, \gamma^n + \left(\sum_{t \geq 0} \frac{g\left(X^{N_m+1+t}\right) \gamma^t}{\binom{N_m+1+t}{t}} \right) * \gamma^{N_m+1} \end{aligned}$$

and set $h := \sum_{t \geq 1} \frac{g\left(X^{N_m+1+t}\right)\gamma^t}{\binom{N_m+1+t}{t}}$. We may now compute

$$\begin{aligned} g \cdot m &= \left(\sum_{n=0}^{N_m} g(X^n)\, \gamma^n + h * \gamma^{N_m+1} \right) \cdot m \\ &= \sum_{i=0}^{N_m} g\left(X^i\right) \gamma^i \cdot m + h \cdot \left(\gamma^{N_m+1} \cdot m \right) \\ &= \sum_{i=0}^{N_m} g\left(X^i\right) \gamma^i \cdot m = \sum_{i=0}^{N_m} \left(\gamma^i \cdot m \right) g\left(X^i\right), \end{aligned}$$

which means exactly that M is rational.

Exercise 2.22. Show that $(\Bbbk[X]^*, *)$ is not a rational $\Bbbk[X]^*$-module.

Solution 1. As we have seen, $\Bbbk[X]^* \cong \Bbbk[[Z]]$. Therefore, $\gamma^t * f = 0$ for some $t \geq 0$ implies that $f = 0$, since $\Bbbk[X]^*$ is a domain.

Exercise 2.23. Consider $C := \Bbbk[X]$ the $\Bbbk$-module of polynomials over a commutative ring $\Bbbk$ with the coalgebra structure given by

$$\Delta\left(X^t\right) = \sum_{i+j=t} X^i \otimes X^j, \qquad \varepsilon\left(X^t\right) = \delta_{0,t}$$

for all $t \in \mathbb{N}$. For all $s \in \mathbb{N}$, consider $\gamma^s : \Bbbk[X] \to \Bbbk : X^i \mapsto \delta_{i,s}$. Prove that a C^*-module M is rational if and only if for every $m \in M$ there exists $t_m \in \mathbb{N}$ such that $\gamma^{t_m} \cdot m = 0$.

3. Integral theory

Integrals are an important tool in the theory of Hopf algebras. From the geometric point of view, they correspond to the existence of a Haar measure on the associated topological group in the sense of example 3.3 and, in general, they can be used to discuss when a Hopf algebra is Frobenius, separable or semisimple.

Definition 3.1. A $\Bbbk$-algebra A is said to be *augmented* if it comes together with an algebra morphism $\varepsilon : A \to \Bbbk$. Given an augmented $\Bbbk$-algebra (A, ε), we say that an element $t \in A$ is a *left integral in* A if for all $a \in A$ we have $at = \varepsilon(a)\, t$. It is a *right integral* if $ta = t\varepsilon(a)$ instead. We denote by $\int_A^l$ and $\int_A^r$ the modules of left and right integrals in A, respectively.

Notice that if B is a bialgebra or a Hopf algebra, then both B and B^* are augmented algebras.

Definition 3.2. A left (resp. right) integral in B^* is called a *left* (resp. *right*) *integral on* B.

Example 3.3. The following example is intended to explain the terminology "*integral*". Recall that a topological group (G, τ, m, e) is a topological space (G, τ) together with a group structure (G, m, e) such that $m : G \times G \to G : (g, g') \mapsto gg'$ and $\mathsf{inv} : G \to G : g \mapsto g^{-1}$ are continuous ($G \times G$ has the product topology). For example, the group $\mathsf{GL}_2(\mathbb{R})$ of 2×2 invertible matrices over the reals with the Euclidean topology induced by the inclusion $\mathsf{GL}_2(\mathbb{R}) \subseteq \mathbb{R}^4$ is a topological group (the multiplication is made by polynomial entries). Denote by $\mathcal{R}(G)$ the vector subspace of $(\mathbb{R}G)^\circ$ containing those functions which are also continuous (i.e. $\mathcal{R}(G) = (\mathbb{R}G)^\circ \cap \mathcal{C}^0(G)$). It can be shown that $\mathcal{R}(G)$ is a Hopf algebra, called *the Hopf algebra of continuous representative functions* on G. If G is also compact as a topological space, then there exists always a measure ν on G, the so-called *Haar measure*. If we set

$$\sigma(f) := \int_G f \mathsf{d}\nu$$

then $\sigma \in \mathcal{R}(G)^*$ is an integral on $\mathcal{R}(G)$ in the above sense (see [1, Example 3.14], [5, §5.1]). It also satisfies the additional condition that $\sigma(f^2) > 0$ for every $f \neq 0$. It is possible to show that, in fact, this is part of an anti-equivalence of categories between the category of compact topological groups and the category of real commutative Hopf algebras admitting an integral σ as above (plus an additional technical condition ensuring that the grouplikes of the finite dual *separate elements*, in the sense that if x and y are different in H Hopf, then there exists $f \in G(H^\circ)$ such that $f(x) \neq f(y)$).

Now, let us go back to the main track. Assume that B is free as a $\Bbbk$-module and recall that B^* is in particular a left B^*-module with obvious action given through $*$. We may consider its rational part, $B^{*\mathsf{rat}}$, which is now a right B-comodule with coaction $\delta(f) = \sum f_{[0]} \otimes f_{[1]}$ uniquely determined by the relation $g * f = \sum f_{[0]} g\left(f_{[1]}\right)$ for every $g \in B^*$.

Lemma 3.4. *Left integrals on B form a two-sided ideal in B^*. Moreover, if B is free as a $\Bbbk$-module then $\int_{B^*}^l$ is rational as left B^*-module and $\int_{B^*}^l = (B^{*\mathsf{rat}})^{\mathsf{co}B}$.*

Proof. Recall that B^* is augmented with augmentation $\varepsilon_* : B^* \to \Bbbk : f \mapsto f(1)$. If λ is a left integral and $g \in B^*$, then $\lambda * g$ is still a left integral because

$$f * (\lambda * g) = (f * \lambda) * g = f(1)\,\lambda * g$$

for all $f \in B^*$. Of course, $g * \lambda = g(1)\,\lambda$ is still an integral, whence $\int_{B^*}^l$ is a two-sided ideal of B^*. Assume that B is free as a $\Bbbk$-module. Since $f * \lambda = f(1)\,\lambda$ for all $f \in B^*$, we have that $\delta : \int_{B^*}^l \to \int_{B^*}^l \otimes B : \lambda \mapsto \lambda \otimes 1$ satisfies the definition of rational module and we have $\int_{B^*}^l \subseteq B^{*\mathsf{rat}}$. In addition, we have in fact that $\int_{B^*}^l \subseteq \left(B^{*\mathsf{rat}}\right)^{\mathsf{co}B}$, since δ coincides with the restriction of the coaction of $B^{*\mathsf{rat}}$ to $\int_{B^*}^l$. Conversely, if $f \in \left(B^{*\mathsf{rat}}\right)^{\mathsf{co}B}$ then for every $g \in B^*$ we have $g * f = \sum f_0 g(f_1) = g(1)\,f$ and hence $f \in \int_{B^*}^l$, as claimed. $\square$

Lemma 3.5. *Assume that B is free or finitely generated and projective over $\Bbbk$. An element $\lambda \in B^*$ is a left integral on B if and only if $\sum b_{(1)}\lambda\left(b_{(2)}\right) = \lambda(b)\,1$ for all $b \in B$.*

Proof. Clearly, $f * \lambda = \varepsilon_*(f)\,\lambda$ for all $f \in B^*$ if and only if $\sum f\left(b_{(1)}\right)\lambda\left(b_{(2)}\right) = f(1)\,\lambda(b)$ for all $b \in B$ and $f \in B^*$, if and only if $f\left(\sum b_{(1)}\lambda\left(b_{(2)}\right)\right) = f(1\lambda(b))$. By the usual dual basis trick, this holds if and only if $\sum b_{(1)}\lambda\left(b_{(2)}\right) = \lambda(b)\,1$ for all $b \in B$. $\square$

If in addition B is a Hopf algebra with antipode S, then we can introduce the following right B-action on B^*

$$(f \leftharpoondown b)(a) := f(aS(b))$$

for every $a, b \in B$ and $f \in B^*$. Associativity and unitality follow from anti-multiplicativity and unitality of S.

Theorem 3.6. *Assume that H is a Hopf algebra which is free as a $\Bbbk$-module. Then $H^{*\mathsf{rat}}$ is a Hopf module with structures*

$$\begin{aligned} \mu &: H^{*\mathsf{rat}} \otimes H \to H^{*\mathsf{rat}}, & f \otimes b &\mapsto (f \leftharpoondown b), \\ \delta &: H^{*\mathsf{rat}} \to H^{*\mathsf{rat}} \otimes H, & f &\mapsto \sum f_{[0]} \otimes f_{[1]}. \end{aligned}$$

Proof. We already know that δ makes of $H^{*\mathsf{rat}}$ a H-comodule. The proof of the fact that the action is well-defined will be performed by resorting to a smart trick. Recall from Lemma 2.12 that $f \leftharpoondown b \in H^{*\mathsf{rat}}$ if and only if there exists $\sum g_i \otimes b_i \in H^* \otimes H$ such that $h * (f \leftharpoondown b) = \sum g_i h(b_i)$ for all $h \in H^*$. If such an element exists, then we put $\delta(f \leftharpoondown b) = \sum g_i \otimes b_i$. However, here we are claiming that $H^{*\mathsf{rat}}$ is going to be not only a H-module but a Hopf module, whence we may more easily check that the element

$$\sum f_{[0]} \leftharpoondown b_{(1)} \otimes f_{[1]}b_{(2)} \in H^* \otimes H$$

satisfies the condition and this will tell us at the same time that $f \leftharpoondown b \in H^{*\mathsf{rat}}$ and that this action is compatible with the coaction. Therefore, denote by $\rightharpoonup$ the usual left H-action on H^*, that is to say, $(b \rightharpoonup f)(a) = f(ab)$ for all $a, b \in H$ and $f \in H^*$. For every $h \in H^*$ let us compute

$$\left(\sum f_{[0]} \leftharpoondown b_{(1)} h\left(f_{[1]}b_{(2)}\right)\right)(a) = \sum f_{[0]}\left(aS\left(b_{(1)}\right)\right) h\left(f_{[1]}b_{(2)}\right)$$

$$
\begin{aligned}
&= \sum f_{[0]} \left(aS \left(b_{(1)} \right) \right) \left(b_{(2)} \rightharpoonup h \right) \left(f_{[1]} \right) \\
&= \sum \left(\left(b_{(2)} \rightharpoonup h \right) * f \right) \left(aS \left(b_{(1)} \right) \right) \\
&= \sum \left(b_{(2)} \rightharpoonup h \right) \left(a_{(1)} S \left(b_{(1)} \right)_{(1)} \right) f \left(a_{(2)} S \left(b_{(1)} \right)_{(2)} \right) \\
&= \sum \left(b_{(3)} \rightharpoonup h \right) \left(a_{(1)} S \left(b_{(2)} \right) \right) f \left(a_{(2)} S \left(b_{(1)} \right) \right) \\
&= \sum h \left(a_{(1)} S \left(b_{(2)} \right) b_3 \right) f \left(a_{(2)} S \left(b_{(1)} \right) \right) \\
&= \sum h \left(a_{(1)} \right) f \left(a_{(2)} S \left(b \right) \right) \\
&= \sum h \left(a_{(1)} \right) \left(f \leftharpoondown b \right) \left(a_{(2)} \right) = \left(h * \left(f \leftharpoondown b \right) \right) (a)
\end{aligned}
$$

for all $a \in H$, whence $\sum f_{[0]} \leftharpoondown b_{(1)} h \left(f_{[1]} b_{(2)} \right) = h * (f \leftharpoondown b)$ for all $h \in H^*$ and we are done. □

As a consequence, we may apply the Structure Theorem for Hopf modules to claim that for every free Hopf algebra H there exists an isomorphism of Hopf modules

$$
\begin{aligned}
\vartheta : \textstyle\int_{H^*}^{l} \otimes H &\longleftrightarrow H^{*\mathsf{rat}} \\
\lambda \otimes b &\longmapsto (\lambda \leftharpoondown b) \\
\sum f_{[0]} S \left(f_{[1]} \right) \otimes f_{[2]} &\longleftarrow f
\end{aligned}
\tag{5}
$$

Notice that this does not necessarily imply that $\int_{B^*}^{l} \neq 0$, as it is not necessarily true that $H^{*\mathsf{rat}} \neq 0$. The following examples will clarify the situation.

Example 3.7. Let $(G, \cdot, 1)$ be a group and consider the group algebra $H := \Bbbk G = \bigoplus_{g \in G} \Bbbk e_g$. This is a Hopf algebra (free over $\Bbbk$) with

$$\Delta(e_g) = e_g \otimes e_g, \qquad \varepsilon(e_g) = 1, \qquad S(e_g) = e_{g^{-1}}$$

for every $g \in G$. Consider the $\Bbbk$-linear map $e_1^* : H \to \Bbbk$ satisfying $e_1^*(e_g) = \delta_{1,g}$. Let us show that $\int_{H^*}^{l} = \Bbbk e_1^*$. On the one hand, for every $f \in H^*$ and $g \in G$ we have

$$(f * e_1^*)(e_g) = f(e_g) e_1^*(e_g) = f(e_g) \delta_{1,g} = f(e_1) \delta_{1,g} = f(e_1) e_1^*(e_g) = (f(e_1) e_1^*)(e_g)$$

and hence $f * e_1^* = f(e_1) e_1^* = \varepsilon_*(f) e_1^*$. On the other hand, in light of Lemma 3.5, for every $\lambda \in \int_{H^*}^{l}$ we have that

$$e_g \lambda(e_g) = \sum e_{g_1} \lambda(e_{g_2}) = 1 \lambda(e_g) = e_1 \lambda(e_g).$$

If $g \neq 1$ then e_g and e_1 are linearly independent and hence $\lambda(e_g) = 0$. As a consequence,

$$\lambda(e_g) = \lambda(e_g) \delta_{1,g} = \lambda(e_1) \delta_{1,g} = (\lambda(e_1) e_1^*)(e_g)$$

for every $g \neq 1$. It clearly holds for $g = 1$ as well, whence $\lambda = \lambda(e_1) e_1^*$.

Example 3.8. Let $\Bbbk$ be a field of characteristic 0 and $H := \Bbbk[X]$ be the polynomial algebra with Hopf algebra structure given by

$$\Delta(X) = X \otimes 1 + 1 \otimes X, \qquad \varepsilon(X) = 0, \qquad S(X) = -X.$$

Let us show that $\int_{B^*}^{l} = 0$. Consider the following assignment

$$\phi : H^* \to \Bbbk[[Z]] : f \mapsto \sum_{n \geq 0} \frac{f(X^n)}{n!} Z^n.$$

This is an isomorphism of $\Bbbk$-algebras with inverse given by

$$\phi^{-1} : \Bbbk[[Z]] \to H^* : \sum_{n \geq 0} k_n Z^n \mapsto [X^n \mapsto k_n].$$

Multiplicativity can be checked directly:

$$\begin{aligned}
\phi(f * g) &= \sum_{n \geq 0} \frac{(f * g)(X^n)}{n!} Z^n = \sum_{n \geq 0} \left(\sum_{i+j=n} \binom{n}{i} \frac{f(X^i)\, g(X^j)}{n!} \right) Z^n \\
&= \sum_{n \geq 0} \left(\sum_{i+j=n} \frac{n!}{i!j!} \frac{f(X^i)\, g(X^j)}{n!} \right) Z^n = \sum_{n \geq 0} \left(\sum_{i+j=n} \frac{f(X^i)}{i!} \frac{g(X^j)}{j!} \right) Z^n \\
&= \left(\sum_{i \geq 0} \frac{f(X^i)}{i!} Z^i \right) \left(\sum_{j \geq 0} \frac{g(X^j)}{j!} Z^j \right) = \phi(f)\, \phi(g).
\end{aligned}$$

Let now λ be a left integral on H, i.e. $\lambda \in \int_{B^*}^{l}$. For every $f \in H^*$ we have $f * \lambda = f(1)\lambda$ and, by applying ϕ to both sides,

$$\phi(f)\, \phi(\lambda) = f(1)\, \phi(\lambda) = \mathsf{ev}_0(\phi(f))\, \phi(\lambda).$$

If we observe that $\Bbbk[[Z]]$ is an augmented algebra with augmentation $\mathsf{ev}_0 : \Bbbk[[Z]] \to \Bbbk$, then $\phi(\lambda)$ is a left integral in $\Bbbk[[Z]]$ (morphisms of augmented algebras send integrals to integrals). In particular,

$$Z\phi(\lambda) = \mathsf{ev}_0(Z)\, \phi(\lambda) = 0$$

from which we get that $\phi(\lambda) = 0$, because $\Bbbk[[Z]]$ is an integral domain, and so $\lambda = 0$.

Example 3.9. Let $(G, \cdot, 1)$ be a finite group and consider the element $\Lambda := \sum_{g \in G} e_g \in \Bbbk G$. For every $h \in G$ we have

$$e_h \Lambda = \sum_{g \in G} e_h e_g = \sum_{g \in G} e_{h \cdot g}.$$

Since G is a group, the assignment $G \to G : g \mapsto h \cdot g$ is bijective with inverse $G \to G : g \mapsto h^{-1} \cdot g$. Therefore, as sets, $\{e_{h \cdot g} \mid g \in G\} = \{e_g \mid g \in G\}$ and hence

$$e_h \Lambda = \Lambda = \varepsilon(e_h)\Lambda.$$

This proves that Λ is a left integral in $\Bbbk G$. The same proof works to show that Λ is also a right integral. Let Λ' be any other left integral in $\Bbbk G$. Observe that

$$\varepsilon(\Lambda)\, \Lambda' = \Lambda \Lambda' = \varepsilon(\Lambda')\, \Lambda$$

and that $\varepsilon(\Lambda) = |G|$. If $\Bbbk$ is a field whose characteristic does not divide $|G|$, then $\varepsilon(\Lambda) \neq 0$ and hence $\varepsilon(\Lambda') \neq 0$ as well. We may also substitute Λ with $\tilde{\Lambda} := \Lambda/\varepsilon(\Lambda)$, so that $\varepsilon\left(\tilde{\Lambda}\right) = 1$ and so

$$\Lambda' = \varepsilon\left(\tilde{\Lambda}\right)\Lambda' = \varepsilon(\Lambda')\,\tilde{\Lambda} \in \Bbbk\tilde{\Lambda}.$$

The foregoing examples are not restrictive. The following result (whose proof is omitted) ensures that integrals are unique, at least over a field. A proof can be found in [1, Theorem 3.3.10], based on the paper [10].

Theorem 3.10. *(*The uniqueness of integrals*) Let $\Bbbk$ be a field and H a Hopf $\Bbbk$-algebra. Then*

$$\dim_{\Bbbk}\left(\int_{H^*}^{l}\right) \leq 1.$$

4. Integrals and finite Hopf algebras

4.1. **Existence of integrals.** Let us begin by proving the existence of integrals for finitely generated and projective Hopf algebras.

Theorem 4.1. *(*[12, Proposition 1.1]*) Let H be a finitely generated and projective Hopf algebra with dual basis $\sum_i e_i \otimes e_i^* \in H \otimes H^*$. Then for every $h \in H$ the element*

$$t_h := \sum_i e_{i_{(1)}} e_i^*\left(S^2\left(e_{i_{(2)}}\right)h\right)$$

is a left integral in H. Moreover, at least one t_h is non-zero.

Proof. For every $a \in H$ let us compute directly

$$\begin{aligned}
\varepsilon(a)\,t_h &= \sum_i e_{i_{(1)}} e_i^*\left(\varepsilon(a)\,S^2\left(e_{i_{(2)}}\right)h\right) = \sum_i e_{i_{(1)}} e_i^*\left(a_{(1)}S\left(a_{(2)}\right)S^2\left(e_{i_{(2)}}\right)h\right) \\
&= \sum_{i,j} e_{i_{(1)}} e_i^*\left(a_{(1)}e_j\right) e_j^*\left(S\left(a_{(2)}\right)S^2\left(e_{i_{(2)}}\right)h\right) \\
&= \sum_j \left(a_{(1)}e_j\right)_{(1)} e_j^*\left(S\left(a_{(2)}\right)S^2\left(\left(a_{(1)}e_j\right)_{(2)}\right)h\right) \\
&= \sum_j a_{(1)}e_{j_{(1)}} e_j^*\left(S\left(a_{(3)}\right)S^2\left(a_{(2)}e_{j_{(2)}}\right)h\right) \\
&= \sum_j a_{(1)}e_{j_{(1)}} e_j^*\left(S\left(a_{(3)}\right)S^2\left(a_{(2)}\right)S^2\left(e_{j_{(2)}}\right)h\right) \\
&= \sum_j a_{(1)}e_{j_{(1)}} e_j^*\left(S\left(S\left(a_{(2)}\right)a_{(3)}\right)S^2\left(e_{j_{(2)}}\right)h\right) \\
&= \sum_j a e_{j_{(1)}} e_j^*\left(S^2\left(e_{j_{(2)}}\right)h\right) = a t_h.
\end{aligned}$$

This proves that $t_h \in \int_H^l$ for every $h \in H$. Moreover,

$$\sum_k e_k^*\left(S\left(t_{e_k}\right)\right) = \sum_k e_k^*\left(S\left(\sum_i e_{i_{(1)}} e_i^*\left(S^2\left(e_{i_{(2)}}\right)e_k\right)\right)\right)$$

$$
\begin{aligned}
&= \sum_{k,i} e_i^* \left(S^2\left(e_{i_{(2)}}\right) e_k\right) e_k^* \left(S\left(e_{i_{(1)}}\right)\right) \\
&= \sum_{i} e_i^* \left(S^2\left(e_{i_{(2)}}\right) S\left(e_{i_{(1)}}\right)\right) \\
&= \sum_{i} e_i^* \left(S\left(e_{i_{(1)}} S\left(e_{i_{(2)}}\right)\right)\right) \\
&= \sum_{i} \varepsilon\left(e_i\right) e_i^* \left(S\left(1\right)\right) = \varepsilon\left(S\left(1\right)\right) = 1
\end{aligned}
$$

whence at least one t_{e_k} has to be non-zero. □

Something more precise can be said if we are working over a field. Recall that if H is a Hopf algebra which is finitely generated and projective over the commutative ring $\Bbbk$, then we already know that every H^*-module is rational. In particular, $H^{*\mathsf{rat}} = H^*$ and Equation (5) becomes

$$\int_{H^*}^{l} \otimes H \cong H^*.$$

Lemma 4.2. *For a finite-dimensional Hopf algebra H over a field $\Bbbk$ there exists a non-zero integral λ on H such that $\int_{H^*}^{l} = \Bbbk\lambda$.*

Proof. A finite-dimensional Hopf algebra is in particular finitely generated and projective, whence we have $\int_{H^*}^{l} \otimes H \cong H^*$ as Hopf H-modules. The latter isomorphism is in particular of $\Bbbk$-vector spaces, whence by comparing the dimensions ($\dim_{\Bbbk}(H) = \dim_{\Bbbk}(H^*)$), we conclude that $\dim_{\Bbbk}\left(\int_{H^*}^{l}\right) = 1$. □

Remark 4.3. Lemma 4.2 allows us to give a more conceptual proof of the fact that the space of integrals in the group algebra $\Bbbk G$ over a finite group G is one-dimensional, which does not use the hypothesis on the characteristic of the field $\Bbbk$. Assume that $f : H \to H'$ is a surjective Hopf algebra map and let $t \in \int_{H}^{l}$. For every $h' \in H'$ there exists $h \in H$ such that $h' = f(h)$ and hence $h'f(t) = f(h)f(t) = f(ht) = \varepsilon(h)f(t) = \varepsilon(f(h))f(t) = \varepsilon(h')f(t)$. This shows that $f(t) \in \int_{H'}^{l}$, so that f induces $\int f : \int_{H}^{l} \to \int_{H'}^{l}$. Let $H = \Bbbk G$. Since H is finite-dimensional, we have an isomorphism of Hopf algebras $H \cong H^{**}$. This entails that $\int_{H}^{l} \cong \int_{H^{**}}^{l}$. Since H^* is a Hopf algebra, Lemma 4.2 states exactly that $\int_{H^{**}}^{l} = \Bbbk\lambda$ for some integral $\lambda : H^* \to \Bbbk$.

4.2. **Integrals, semisimplicity and separability.** Let A be a $\Bbbk$-algebra. A *Casimir element* is an element $\sum e' \otimes e'' \in A \otimes A$ such that $\sum ae' \otimes e'' = \sum e' \otimes e''a$ for every $a \in A$. Denote by $\mathcal{C}_A$ the $\Bbbk$-module of all the Casimir elements. The algebra A is called *separable* if there exists $e \in \mathcal{C}_A$ such that $\sum e'e'' = 1$. In such a case e is called a *separability idempotent* (it is idempotent in $A \otimes A^{\mathsf{op}}$):

$$
\begin{aligned}
e^2 &= \left(\sum e' \otimes e''\right)\left(\sum f' \otimes f''\right) = \sum e'f' \otimes e'' \cdot^{\mathsf{op}} f'' = \sum e'f' \otimes f''e'' \\
&= \sum e'e''f' \otimes f'' = \sum f' \otimes f'' = e.
\end{aligned}
$$

We are going to see how integrals can be used to determine when a Hopf algebra is semisimple or separable.

Lemma 4.4. *Let H be a Hopf algebra. We have the following maps*

$$p:\mathcal{C}_H\to\int_H^l : e\mapsto e'\varepsilon\left(e''\right), \qquad p':\mathcal{C}_H\to\int_H^r e\mapsto \varepsilon\left(e'\right)e'',$$

$$i:\int_H^l\to\mathcal{C}_H : t\mapsto\sum t_{(1)}\otimes S\left(t_{(2)}\right), \qquad i':\int_H^r\to\mathcal{C}_H : t\mapsto\sum S\left(t_{(1)}\right)\otimes t_{(2)},$$

satisfying $p\circ i=Id$ and $p'\circ i'=Id$.

Proof. We only prove that p and i are well-defined. The other checks are analogous. First of all, for every $h\in H$ we have

$$he'\varepsilon\left(e''\right)=e'\varepsilon\left(e''h\right)=\varepsilon\left(h\right)e'\varepsilon\left(e''\right),$$

whence $p\left(e\right)$ is a left integral for every $e\in\mathcal{C}_H$. Secondly,

$$\begin{aligned}\sum ht_{(1)}\otimes S\left(t_{(2)}\right)&=\sum h_{(1)}t_{(1)}\otimes S\left(t_{(2)}\right)S\left(h_{(2)}\right)h_{(3)}=\sum h_{(1)}t_{(1)}\otimes S\left(h_{(2)}t_{(2)}\right)h_{(3)}\\&=\sum\left(h_{(1)}t\right)_{(1)}\otimes S\left(\left(h_{(1)}t\right)_{(2)}\right)h_{(2)}=\sum t_{(1)}\otimes S\left(t_{(2)}\right)h,\end{aligned}$$

whence $i\left(t\right)\in\mathcal{C}_H$ for every $t\in\int_H^l$. ∎

Recall that a ring R is *semisimple* if every surjective morphism $f:M\to N$ of left (equivalently, right) R-modules splits, that is to say, there exists a morphism $\sigma:N\to M$ of left R-modules such that $f\circ\sigma=Id_N$.

Theorem 4.5. *(*Maschke Theorem for Hopf algebras*) For a Hopf algebra over a field $\Bbbk$ the following assertions are equivalent.*

(a) *H is semisimple as a ring.*
(b) *There exists $t\in\int_H^l$ such that $\varepsilon\left(t\right)=1$.*
(c) *H is separable as an algebra.*

Proof. To prove that (a) implies (b) consider the left H-linear morphism $\varepsilon:H\to\Bbbk$. Since H is semisimple and ε is surjective, it admits a section left H-linear $\sigma:\Bbbk\to H$. Set $t:=\sigma\left(1_\Bbbk\right)$ and observe that for every $h\in H$ we have $ht=h\sigma\left(1_\Bbbk\right)=\sigma\left(h\cdot 1_\Bbbk\right)=\sigma\left(\varepsilon\left(h\right)1_\Bbbk\right)=\varepsilon\left(h\right)t$ and that $\varepsilon\left(t\right)=\varepsilon\left(\sigma\left(1_\Bbbk\right)\right)=1_\Bbbk$.

To prove that (b) implies (c) consider the Casimir element $e=i\left(t\right)=\sum t_{(1)}\otimes S\left(t_{(2)}\right)$. Of course, $\sum t_{(1)}S\left(t_{(2)}\right)=\varepsilon\left(t\right)1_B=1_B$, whence e is a separability idempotent.

Finally, to prove that (c) implies (a) let us proceed as follows. Pick any surjective morphism of left H-modules $\pi:M\to N$. Since it is in particular of $\Bbbk$-vector spaces it admits a $\Bbbk$-linear section $\sigma:N\to M$. Of course, σ is not H-linear in general, but we may consider $\tau:N\to M:n\longmapsto\sum e'\sigma\left(e''n\right)$. This is H-linear because $\sum e'\sigma\left(e''hn\right)=\sum he'\sigma\left(e''n\right)$ for every $h\in H$ and it is still a section since $\pi\left(\tau\left(n\right)\right)=\sum\pi\left(e'\sigma\left(e''n\right)\right)=\sum e'\pi\left(\sigma\left(e''n\right)\right)=\sum e'e''n=n$ for every $n\in N$. ∎

Remark 4.6. An integral t such that $\varepsilon\left(t\right)=1$ is called a *total integral.* Notice that if t is a left integral for a Hopf algebra H then

$$S\left(t\right)h=\sum S\left(h_{(1)}t\right)h_{(2)}=\sum S\left(t\right)S\left(h_{(1)}\right)h_{(2)}=S\left(t\right)\varepsilon\left(h\right),$$

so that $S(t)$ is a right integral. It makes no sense to distinguish between left total integrals and right total integrals because

$$t=\varepsilon(t)\,t=\varepsilon(S(t))\,t=S(t)\,t=S(t)\,\varepsilon(t)=S(t)\,,$$

so that they are both left and right integrals at the same time.

Theorem 4.7. *(Villamayor-Zelinsky, 1966) A separable algebra over a field $\Bbbk$ is finite-dimensional.*

Proof. Let A be a separable $\Bbbk$-algebra with (possibly infinite) basis $\{e_i \mid i\in I\}$ and let $e=\sum\alpha_k\otimes\beta_k$ be a separability idempotent. For every $i\in I$ define $e_i^*:A\to\Bbbk$ by setting $e_i^*(e_j)=\delta_{i,j}$. Then, given any $a\in A$ we may write $a=\sum_{i\in I(a)}a_ie_i$ with $I(a)$ finite and $e_k^*(a)=e_k^*\left(\sum_{i\in I(a)}a_ie_i\right)=\sum_{i\in I(a)}a_ie_k^*(e_i)=a_k$. Therefore, $a=\sum_{i\in I(a)}e_i^*(a)\,e_i$. Now, let us compute

$$a=a1_A=\sum_k a\alpha_k\beta_k=\sum_k a\alpha_k\left(\sum_{i\in I(\beta_k)}e_i^*(\beta_k)\,e_i\right)=\sum_{k,i\in I(\beta_k)}\alpha_ke_ie_i^*(\beta_ka)\,.$$

This entails that a lives in the finite-dimensional subspace generated by the α_ke_i's. □

Corollary 4.8. *A Hopf algebra over a field $\Bbbk$ satisfying any one of the equivalent conditions of Theorem 4.5 is finite-dimensional.*

Remark 4.9. Let us highlight some facts concerning Theorem 4.5.

(1) In light of Remark 4.6, in the proof of (2) ⇒ (3) one could have considered $e=i'(t)=\sum S\left(t_{(1)}\right)\otimes t_{(2)}$ as well.
(2) The equivalence between (2) and (3) holds even when $\Bbbk$ is simply a commutative ring. Indeed, the implication from (2) to (3) is proved exactly as above and to prove that (3) implies (2) one observes that $t=p(e)=\sum e'\varepsilon(e'')\in\int_H^l$ satisfies $\varepsilon(t)=\sum\varepsilon(e'\varepsilon(e''))=\varepsilon(\sum e'e'')=1_\Bbbk$.
(3) Every separable algebra over a field is semisimple, by the same proof of (3) ⇒ (1). However, the converse is not true in general: the real numbers $\mathbb{R}$ form a semisimple ring (since they form a field), which is not separable as a $\mathbb{Q}$-algebra because is not finite-dimensional. Even if A is a finite-dimensional semisimple $\Bbbk$-algebra, it may happen that A is not separable.

Exercise 4.10. Let p be a prime number and consider the field $\Bbbk:=\mathbb{Z}_p(Y)$, that is, the field of fractions of $\mathbb{Z}_p[Y]$. Consider the polynomial $q(X):=X^p-Y\in\Bbbk[X]$. It is irreducible in $\Bbbk[X]$ by Eisenstein and Gauss: the ideal $\langle Y\rangle\subseteq\mathbb{Z}_p[Y]$ is prime (in fact, maximal, since the quotient $\mathbb{Z}_p[Y]/\langle Y\rangle\cong\mathbb{Z}_p$ is a field), $Y\in\langle Y\rangle$ and $Y\notin\langle Y^2\rangle$, whence $q(X)$ is irreducible in $(\mathbb{Z}_p[Y])[X]$ by Eisenstein and it is irreducible in $\Bbbk[X]$ by Gauss. The quotient $\Bbbk$-algebra $A:=\Bbbk[X]/\langle p(X)\rangle$ is then a field and as such it is semisimple. However, it is not separable.

Theorem 4.11. *(Maschke Theorem for groups) Let $\Bbbk$ be a field. The following are equivalent for a group G*

(a) *$\Bbbk G$ as a ring is semisimple.*

(b) *G is finite and* $\mathsf{char}\,(\Bbbk) \nmid |G|$.
(c) *$\Bbbk G$ as an algebra is separable.*

Proof. Since $\Bbbk G$ is a Hopf algebra, in light of Theorem 4.5 it will be enough to prove that G is finite and $\mathsf{char}\,(\Bbbk) \nmid |G|$ if and only if $\Bbbk G$ has a total integral. Recall from Example 3.9 that if G is finite and $\mathsf{char}\,(\Bbbk) \nmid |G|$ then $\Lambda := \frac{1}{|G|}\sum_{g\in G} e_g$ is a total integral. Conversely, set $e := \sum_{g\in G} e_g$ and assume that $\Bbbk G$ admits a total integral t. By (b) $\Rightarrow$ (c) in Theorem 4.5 it follows that $\Bbbk G$ is finite-dimensional and hence G is finite. Moreover, $e = e\varepsilon\,(t) = et = \varepsilon\,(e)\,t = |G|\,t$ entails that $|G|$ cannot be 0 as e is not 0. Thus $\mathsf{char}\,(\Bbbk) \nmid |G|$ and the proof is finished. □

Remark 4.12. It may be of interest to note that if the equivalent conditions of Theorem 4.11 then a total integral for $\Bbbk G$ is provided by $\Lambda := \frac{1}{|G|}\sum_{g\in G} e_g$ and a separability idempotent by $e = \frac{1}{|G|}\sum_{g\in G} e_g \otimes e_{g^{-1}}$.

4.3. **Integrals and Frobenius property.** An algebra A is called *Frobenius* if there exists a Casimir element $e \in \mathcal{C}_A$ and a linear operator $\nu \in A^*$ (called the *Frobenius homomorphism*) such that

$$(A \otimes \nu)\,(e) = 1 = (\nu \otimes A)\,(e)\,.$$

We call the pair (ν, e) a *Frobenius system* for A. Recall that for a $\Bbbk$-algebra A, A^* is always an A-bimodule via

$$(a \rightharpoonup f \leftharpoonup b)\,(x) = f\,(bxa)$$

for every $a, b, x \in A$ and $f \in A^*$. Recall also that a bilinear form $\beta : A \times A \to \Bbbk$ is said to be *associative* if $\beta\,(ab, c) = \beta\,(a, bc)$ and it is said to be *non-degenerate* if the assignment $a \mapsto \beta\,(-, a)$ provides an isomorphism $A \to A^*$.

Proposition 4.13. *For a $\Bbbk$-algebra A the following assertions are equivalent*
(a) *A is Frobenius,*
(b) *There exists $e = \sum e' \otimes e'' \in A \otimes A$ and $\nu \in A^*$ such that $a = \sum e'\nu\,(e''a)$ for every $a \in A$,*
(c) *There exists $e = \sum e' \otimes e'' \in A \otimes A$ and $\nu \in A^*$ such that $a = \sum \nu\,(ae')\,e''$ for every $a \in A$,*
(d) *A is finitely generated and projective as a $\Bbbk$-module and $A \cong A^*$ as left A-modules,*
(e) *A is finitely generated and projective as a $\Bbbk$-module and $A \cong A^*$ as right A-modules,*
(f) *A is finitely generated and projective as a $\Bbbk$-module and there exists a non-degenerate bilinear form $\beta : A \times A \to \Bbbk$.*

Proof. To prove that (a) implies (b) notice that for every $a \in A$ we have

$$a = \sum ae'\nu\,(e'') = \sum e'\nu\,(e''a)$$

because e is Casimir. To prove that (b) implies (d) observe that $a = \sum e'\nu\,(e''a)$ for every $a \in A$ entails that A is finitely generated and projective with dual basis $\sum e' \otimes (\nu \leftharpoonup e'')$. In particular, for every $f \in A^*$ we have $f = \sum f\,(e')\,(\nu \leftharpoonup e'')$. Consider

$$\phi : A^* \to A : f \mapsto \sum e'f\,(e'')\,,$$

$$\psi : A \to A^* : b \mapsto [(b \rightharpoonup \nu) : a \mapsto \nu(ab)] .$$

Since for every $a, b, c \in A$ we have

$$\psi(bc)(a) = \nu(abc) = (c \rightharpoonup \nu)(ab) = (b \rightharpoonup (c \rightharpoonup \nu))(a) = (b \rightharpoonup \psi(c))(a),$$

it is clear that ψ is left A-linear. Moreover,

$$a \overset{\psi}{\longmapsto} (a \rightharpoonup \nu) \overset{\phi}{\longmapsto} \sum e'(a \rightharpoonup \nu)(e'') = \sum e'\nu(e''a) = a$$

so that $\phi \circ \psi$ is the identity. To show that this is enough to claim that they are each other inverses, proceed as follows. If $\Bbbk$ is a field, then comparing dimensions allows us to conclude that ψ has to be surjective as well and hence invertible. If $\Bbbk$ is a commutative local ring with unique maximal ideal $\mathfrak{m}$ then consider

$$\begin{aligned} \overline{\phi} : \frac{A^*}{A^*\mathfrak{m}} \to \frac{A}{A\mathfrak{m}}, \qquad & f + A^*\mathfrak{m} \mapsto \sum e' f(e'') + A\mathfrak{m}, \\ \overline{\psi} : \frac{A}{A\mathfrak{m}} \to \frac{A^*}{A^*\mathfrak{m}}, \qquad & a + A\mathfrak{m} \longmapsto (a \rightharpoonup \nu) + A^*\mathfrak{m}. \end{aligned}$$

These are well-defined because ϕ, ψ are $\Bbbk$-linear ($\overline{\psi} = \psi \otimes \frac{\Bbbk}{\mathfrak{m}}$) and $\overline{\phi} \circ \overline{\psi} = \overline{\phi \circ \psi}$ is the identity. Since now $\Bbbk/\mathfrak{m}$ is a field and since $A^*/A^*\mathfrak{m} \cong A^* \otimes_{\Bbbk} \Bbbk/\mathfrak{m} \cong \mathsf{Hom}_{\Bbbk}(A, \Bbbk/\mathfrak{m}) \cong \mathsf{Hom}_{\Bbbk/\mathfrak{m}}(A \otimes \Bbbk/\mathfrak{m}, \Bbbk/\mathfrak{m}) \cong (A/A\mathfrak{m})^*$, we conclude that $\overline{\psi}$ is an isomorphism. Therefore, for every $f \in A^*$ there exists $a \in A$, $g \in A^*$ and $k \in \mathfrak{m}$ such that $f = \psi(a) + gk$, which implies that $[f] = [g]\,k \in X := A^*/\mathrm{im}(\psi)$ and so $X = X\mathfrak{m}$. Since X is finitely generated (because A^* is), Nakayama's lemma implies that $X = 0$ and hence ψ is surjective. Finally, let $\Bbbk$ be commutative and for every prime ideal $\mathfrak{p} \subseteq \Bbbk$ consider

$$\phi_{\mathfrak{p}} : A^*_{\mathfrak{p}} \to A_{\mathfrak{p}} \qquad \text{and} \qquad \psi_{\mathfrak{p}} : A_{\mathfrak{p}} \to A^*_{\mathfrak{p}}$$

the localizations at $\mathfrak{p}$ of ϕ and ψ. Since $\phi_{\mathfrak{p}} \circ \psi_{\mathfrak{p}} = (\phi \circ \psi)_{\mathfrak{p}}$ is the identity, we conclude that $\psi_{\mathfrak{p}}$ is an isomorphism for every prime ideal $\mathfrak{p}$ and hence ψ is an isomorphism of left A-modules with inverse ϕ.

To prove that (d) implies (f) consider a left A-linear isomorphism $\psi : A \to A^*$ and set $\nu := \psi(1)$. Since ψ is A-linear, $\psi(a) = a \rightharpoonup \nu$ for every $a \in A$. If we consider the assignment

$$\beta : A \times A \to \Bbbk : (a, b) \mapsto \nu(ab),$$

then it obviously provides a bilinear form such that $\beta(ab, c) = \nu((ab)c) = \nu(a(bc)) = \beta(a, bc)$, whence it is associative. Moreover, the assignment $a \mapsto \beta(-, a) = a \rightharpoonup \nu$ coincides with ψ and hence it is an isomorphism, so that β is non-degenerate.

To prove that (f) implies (d) consider the $\Bbbk$-linear isomorphism $\psi : A \to A^* : a \mapsto \beta(-, a)$. Since $\psi(ab)(x) = \beta(x, ab) = \beta(xa, b) = \psi(b)(xa) = (a \rightharpoonup \psi(b))(x)$ for every $a, b, x \in A$, we conclude that ψ is also left A-linear.

Finally, to prove that (d) implies (a) suppose that $\phi : A^* \to A$ is a left A-linear isomorphism with inverse ψ. Let $\{e_i, e_i^*\}$ be a dual basis of A and set $y_i := \phi(e_i^*)$, $\nu = \psi(1)$. Then, $\psi(a) = a \rightharpoonup \psi(1) = a \rightharpoonup \nu$ for all $a \in A$ and $e_i^* = \psi(y_i) = y_i \rightharpoonup \nu$, whence

$$a = \sum e_i^*(a)\, e_i = \sum (y_i \rightharpoonup \nu)(a)\, e_i = \sum \nu(a y_i)\, e_i.$$

For $a = 1$ we get $1 = \sum \nu(y_i) e_i$. On the other hand

$$\nu = \sum e_i^* \nu(e_i) = \sum (y_i \rightharpoonup \nu) \nu(e_i) = \left(\sum y_i \nu(e_i)\right) \rightharpoonup \nu$$

from which it follows that

$$1 = \phi(\nu) = \phi\left(\left(\sum y_i \nu(e_i)\right) \rightharpoonup \nu\right) = \sum y_i \nu(e_i) \phi(\nu) = \sum y_i \nu(e_i).$$

We are left to prove that $\sum y_i \otimes e_i$ is a Casimir element. To this aim, compute

$$\begin{aligned} \sum y_i \otimes e_i a &= \sum \phi(e_i^*) \otimes e_k^*(e_i a) e_k = \sum \phi(e_i^* e_k^*(e_i a)) \otimes e_k \\ &= \sum \phi(e_i^*(a \rightharpoonup e_k^*)(e_i)) \otimes e_k = \sum \phi(a \rightharpoonup e_k^*) \otimes e_k \\ &= \sum (a \rightharpoonup \phi(e_k^*)) \otimes e_k = \sum (a \rightharpoonup y_k) \otimes e_k. \end{aligned}$$

Summing up, we showed that (d) $\iff$ (f) and that (a) $\Longrightarrow$ (b) $\Longrightarrow$ (d) $\Longrightarrow$ (a), proving in this way that (a) $\iff$ (b) $\iff$ (d) $\iff$ (f). The proof that (a) $\iff$ (c) $\iff$ (e) $\iff$ (f) is completely analogous. $\square$

Remark 4.14. By analizing closely the chain of implications (2) $\Longrightarrow$ (4) $\Longrightarrow$ (1) in the proof of Proposition 4.13, we can see that ν becomes the Frobenius homomorphism and $e = \sum e' \otimes e''$ becomes the Casimir element.

Definition 4.15. For an augmented Frobenius algebra A with Frobenius homomorphism ν, an element $n \in A$ such that $n \rightharpoonup \nu = \varepsilon$ is called a *left norm*. Analogously, an element $N \in A$ such that $\nu \leftharpoonup N = \varepsilon$ is called a *right norm*.

Remark 4.16. Let A be an aumented Frobenius algebra A with Frobenius homomorphism ν and set $\psi : A \to A^* : b \longmapsto (b \rightharpoonup \nu)$. For a left norm n, $n \in \int_B^l$. In fact, for every $a, b \in B$ we have $\nu(ban) = \varepsilon(ba) = \varepsilon(a)\nu(bn) = \nu(b\varepsilon(a)n)$, that is to say, $\psi(an) = \psi(\varepsilon(a)n)$ for every $a \in B$, which implies that $an = \varepsilon(a)n$. Analogously, given a right norm N we can show that $N \in \int_B^r$.

Proposition 4.17. *(*[8, Proposition 3]*) For a finitely generated and projective Hopf* $\Bbbk$*-algebra H, $\int_{H^*}^l$ is a finitely generated and projective* $\Bbbk$*-module of constant rank one.*

Proof. The fact that $\int_{H^*}^l$ is finitely generated and projective follows from the fact that it is a direct summand of H^*, which is finitely generated and projective. Let us prove that it is of rank 1. For every prime ideal $\mathfrak{p}$ of H we have

$$H_{\mathfrak{p}}^* \cong \left(\int_{H^*}^l \otimes_{\Bbbk} H\right)_{\mathfrak{p}} \cong \left(\int_{H^*}^l\right)_{\mathfrak{p}} \otimes_{\Bbbk_{\mathfrak{p}}} H_{\mathfrak{p}}$$

(see [3, Proposition 3.7]). Since all the three $\Bbbk$-modules are finitely generated and projective over $\Bbbk$, their localizations at $\mathfrak{p}$ are finitely generated and projective as $\Bbbk_{\mathfrak{p}}$-modules and hence free of finite rank (because $\Bbbk_{\mathfrak{p}}$ is a local ring).

Since $H_{\mathfrak{p}} \cong H_{\mathfrak{p}}^*$ as free $\Bbbk_{\mathfrak{p}}$-modules of finite rank, it follows that $\left(\int_{H^*}^l\right)_{\mathfrak{p}}$ has to be free of rank one for every prime ideal $\mathfrak{p}$ of H.

$\square$

The following is another useful consequence of the Structure Theorem for Hopf Modules.

Proposition 4.18. *The antipode of a finitely generated and projective Hopf algebra is bijective.*

Proof. Recall that in light of the Structure Theorem we know that

$$\epsilon_{H^*} : \int_{H^*}^{l} \otimes H \to H^* : \sum_i \lambda_i \otimes h_i \longmapsto \sum_i (\lambda_i \leftharpoonup h_i)$$

is an isomorphism of Hopf modules. Assume firstly that $\Bbbk$ is a local ring with unique maximal ideal $\mathfrak{m}$. Then $H^* \cong \int_{H^*}^{l} \otimes H$ implies that $\int_{H^*}^{l}$ is free of rank 1 as a $\Bbbk$-module, generated by some λ. If $h \in \ker(S)$, then $\epsilon_{H^*}(\lambda \otimes h) = \lambda \leftharpoonup h = S(h) \rightharpoonup \lambda = 0$ implies that $\lambda \otimes h = 0$ and hence $h = 0$. Therefore S is injective. To prove that it is also surjective observe that $\overline{H} = H/\mathfrak{m}H$ is a finite-dimensional $\overline{\Bbbk} = \Bbbk/\mathfrak{m}$-vector space and that the Hopf algebra structure passes to the quotient, making of $\overline{H}$ a finite-dimensiomnal $\overline{\Bbbk}$-Hopf algebra. Snake lemma shows that $\overline{S}$ is injective and hence it is bijective by comparing dimensions. In particular, for every $h \in H$ there exists $x, y \in H$ and $k \in \mathfrak{m}$ such that $h = S(x) + ky$. Call $X := H/\mathrm{im}(S)$. In X we have that $[h] = k[y]$, whence $X = \mathfrak{m}X$. Since X is clearly finitely generated, we may apply Nakayama's Lemma to claim that $X = 0$ and hence S is surjective as well.

Let $\Bbbk$ be again a commutative ring. For every prime ideal $\mathfrak{p} \in \Bbbk$, the localization $H_{\mathfrak{p}} \cong \Bbbk_{\mathfrak{p}} \otimes H$ is still a finitely generated and projective Hopf algebra over $\Bbbk_{\mathfrak{p}}$ (localization is an exact functor: if H is direct summand of a free $\Bbbk$-module, $H_{\mathfrak{p}}$ is direct summand of a free $\Bbbk_{\mathfrak{p}}$-module), which now is a local commutative ring. This means that the localization $S_{\mathfrak{p}}$ of the antipode at any prime ideal is bijective. Since being injective, surjective and hence bijective is a local property ([3, Proposition 3.9]), this means that S is bijective as claimed. □

Remark 4.19. For the case of finite-dimensional Hopf algebras over a field. The proof is much easier. Let S be the antipode of H and let $h \in \ker(S)$. By definition, for every $x \in H$ we have

$$(\lambda \leftharpoonup h)(x) = \lambda(xS(h)) = 0,$$

whence $\lambda \leftharpoonup h = 0$. However, ϑ of Equation 5 is an isomorphism and $0 = \lambda \leftharpoonup h = \vartheta(\lambda \otimes h)$ implies that $\lambda \otimes h = 0$ and since $\lambda \neq 0$, we have that $h = 0$. Therefore, S is injective and by comparing dimensions it has to be bijective.

The forthcoming Theorem 4.22 is the main result of this section, but before proving it, we need to prove a technical lemma.

Lemma 4.20. *(*[6]*) The $\Bbbk$-module of left (equivalently, right) integrals in an augmented Frobenius algebra is free of rank one.*

Proof. Let A be a Frobenius augmented algebra with Frobenius morphism ν and consider a left norm $n \in A$. Given a non-zero left integral $t \in A$, note that

$$t \rightharpoonup \nu = \nu(t)\,\varepsilon = \nu(t)\, n \rightharpoonup \nu$$

so that $t = \nu(t)\, n$ by bijectivity of $\psi : A \to A^* : a \longmapsto (a \rightharpoonup \nu)$. This implies that $\int_A^l \subseteq \Bbbk n$ and so $\int_A^l = \Bbbk n$, since n is a left integral in A. Let us prove that the latter is also free as a $\Bbbk$-module. Assume that $k \in \Bbbk$ is such that $kn = 0$. Then $0 = \nu(kn) = k\nu(n) = k$. $\square$

Remark 4.21. Between all left norms, we have a kind of distinguished one. Consider the Frobenius isomorphism $\psi : A \to A^* : a \longmapsto (a \rightharpoonup \nu)$ and consider $n := \psi^{-1}(\varepsilon)$. Then for every $a \in A$ we have $\nu(n) = (\psi^{-1}(\varepsilon) \rightharpoonup \nu)(1) = \varepsilon(1) = 1$ and

$$(n \rightharpoonup \nu)(a) = \nu\left(a\psi^{-1}(\varepsilon)\right) = \nu\left(\psi^{-1}(a \rightharpoonup \varepsilon)\right) = \varepsilon(a)\,\nu\left(\psi^{-1}(\varepsilon)\right) = \varepsilon(a).$$

Theorem 4.22. *([8]) For a bialgebra B the following are equivalent:*

(a) *B is a finitely generated and projective $\Bbbk$-Hopf algebra and $\int_{H^*}^l$ is free of rank one;*
(b) *B is a Frobenius $\Bbbk$-algebra with $\nu \in \int_{H^*}^l$.*

Proof. To prove that (a) implies (b) assume that $\int_{H^*}^l \cong \Bbbk\nu$ for some $\nu \in \int_{B^*}^l$ and recall that we have an B-Hopf module isomorphism

$$\epsilon_{B^*} : \int_{B^*}^l \otimes B \cong B^* : \nu \otimes a \longmapsto (\nu \leftharpoondown a) = (S(a) \rightharpoonup \nu)$$

Since S is invertible (Proposition 4.18), we may consider the $\Bbbk$-linear isomorphism $B \to \int_{B^*}^l \otimes B \to B^* : a \longmapsto \nu \otimes S^{-1}(a) \longmapsto (\nu \leftharpoondown S^{-1}(a)) = (a \rightharpoonup \nu)$ and it is clear that the latter is left B-linear, whence B is Frobenius, and that ν is the Frobenius homomorphism. Conversely, to prove that (b) implies (a) proceed as follows. We already know that B is finitely generated and projective and that we have a left B-linear isomorphism $\psi : B \to B^* : a \longmapsto (a \rightharpoonup \nu)$. Pick a left norm $n \in B$ and consider the assignment

$$s : B \to B : a \longmapsto \sum n_{(1)}\nu\left(an_{(2)}\right).$$

Since both n and ν are left integrals, it follows from Lemma 3.5 that $\sum n_{(1)}\nu\left(n_{(2)}\right) = \nu(n)\, 1 = 1$ and hence that $\sum a_{(1)}s\left(a_{(2)}\right) = \sum a_{(1)}n_{(1)}\nu\left(a_{(2)}n_{(2)}\right) = \varepsilon(a)\sum n_{(1)}\nu\left(n_{(2)}\right) = \varepsilon(a)\, 1$. Thus s is right convolution inverse of the identity. To prove that it is also a left convolution inverse, there are two ways: the first one is to apply [8, Lemma 4] as it is done in [8, Theorem 2] to conclude that it is. Otherwise, notice that $\psi' : B \to B^* : a \longmapsto (\nu \leftharpoonup a)$ is now a right B-linear isomorphism with inverse $f \longmapsto \sum f(e')\, e''$.

Pick a right norm $N \in B$ and consider the map $s' : B \to B : a \longmapsto \sum N_{(1)}\nu\left(N_{(2)}a\right)$. This satisfies

$$\begin{aligned}
s'(s(a)) &= s'\left(\sum n_{(1)}\nu\left(an_{(2)}\right)\right) = \sum N_{(1)}\nu\left(N_{(2)}n_{(1)}\nu\left(an_{(2)}\right)\right) \\
&= \sum N_{(1)}\nu\left(N_{(2)}n_{(1)}\right)\nu\left(an_{(2)}\right) = \sum N_{(1)}a_{(1)}\nu\left(N_{(2)}a_{(2)}n_{(1)}\right)\nu\left(a_{(3)}n_{(2)}\right) \\
&= \sum N_{(1)}a\nu\left(N_{(2)}n_{(1)}\right)\nu\left(n_{(2)}\right) = \sum N_{(1)}\nu\left(N_{(2)}\right)a = a, \\
s(s'(a)) &= s\left(\sum N_{(1)}\nu\left(N_{(2)}a\right)\right) = \sum n_{(1)}\nu\left(N_{(1)}\nu\left(N_{(2)}a\right)n_{(2)}\right) \\
&= \sum n_{(1)}\nu\left(N_{(1)}n_{(2)}\right)\nu\left(N_{(2)}a\right) = \sum a_{(1)}n_{(1)}\nu\left(N_{(1)}a_{(2)}n_{(2)}\right)\nu\left(N_{(2)}a_{(3)}\right)
\end{aligned}$$

$$= \sum an_{(1)}\nu\left(N_{(1)}\nu\left(N_{(2)}\right)n_{(2)}\right) = \sum an_{(1)}\nu\left(n_{(2)}\right) = a,$$

that is to say, s' is the inverse of s. Moreover, observe that

$$\begin{aligned}
(6)\qquad \sum n_{(1)}\nu\left(bn_{(2)}\right)n'_{(1)}\nu\left(an'_{(2)}\right) &= \sum n_{(1)}n'_{(1)}\nu\left(a\nu\left(bn_{(2)}\right)n'_{(2)}\right) \\
&\overset{\left(\nu\in\int^l_{B^*}\right)}{=} \sum n_{(1)}n'_{(1)}\nu\left(ab_{(1)}n_{(2)}\nu\left(b_{(2)}n_{(3)}\right)n'_{(2)}\right) \\
&\overset{\left(n\in\int^l_{B}\right)}{=} \sum n_{(1)}n'_{(1)}\nu\left(ab_{(1)}n_{(2)}n'_{(2)}\right)\nu\left(b_{(2)}n_{(3)}\right) \\
&\overset{\left(n\in\int^l_{B}\right)}{=} \sum n'_{(1)}\nu\left(ab_{(1)}n'_{(2)}\right)\nu\left(b_{(2)}n\right) \\
&= \sum n'_{(1)}\nu\left(abn'_{(2)}\right)\nu\left(n\right) \\
&= \sum n'_{(1)}\nu\left(abn'_{(2)}\right).
\end{aligned}$$

Compute now

$$\begin{aligned}
\sum s\left(a_{(1)}\right)a_{(2)} &= \sum n_{(1)}\nu\left(a_{(1)}n_{(2)}\right)a_{(2)} \\
&= \sum n_{(1)}\nu\left(a_{(1)}n_{(2)}\right)n'_{(1)}\nu\left(N_{(1)}\nu\left(N_{(2)}a_{(2)}\right)n'_{(2)}\right) \\
&\overset{(6)}{=} \sum n_{(1)}\nu\left(N_{(1)}\nu\left(N_{(2)}a_{(2)}\right)a_{(1)}n_{(2)}\right) \\
&= \sum n_{(1)}\nu\left(N_{(1)}a_{(1)}\nu\left(N_{(2)}a_{(2)}\right)n_{(2)}\right) \\
&\overset{\left(N\in\int^r_{B}\right)}{=} \sum n_{(1)}\nu\left(N_{(1)}\nu\left(N_{(2)}\right)n_{(2)}\right)\varepsilon\left(a\right) \\
&\overset{\left(\nu\in\int^l_{B^*}\right)}{=} \sum n_{(1)}\nu\left(n_{(2)}\right)\varepsilon\left(a\right) \\
&\overset{\left(\nu\in\int^l_{B^*}\right)}{=} \varepsilon\left(a\right)1
\end{aligned}$$

and the proof that s is an antipode is complete. We are left to check that $\int^l_{H^*}$ is free of rank one. First of all, recall that H^* is still a Hopf algebra with antipode s^* and that $\nu\in\int^l_{H^*}$. Therefore $i\left(\nu\right) = \sum \nu_{(1)}\otimes s^*\left(\nu_{(2)}\right)$ is a Casimir element for H^*. Observe also that

$$s\left(N\right) = \sum n_{(1)}\nu\left(Nn_{(2)}\right) = \sum n_{(1)}\varepsilon\left(n_{(2)}\right)\nu\left(N\right) = n$$

because N is a right integral. Thus, $\mathsf{ev}_N\in\int^l_{H^{**}}$ in light of Remark 4.3 and

$$\begin{aligned}
\sum \mathsf{ev}_N\left(\nu_{(1)}\right)s^*\left(\nu_{(2)}\right) &= s^*\left(\sum \nu_{(1)}\left(N\right)\nu_{(2)}\right) = s^*\left(\nu\leftharpoonup N\right) = s^*\left(\varepsilon\right) = \varepsilon, \\
\sum \nu_{(1)}\mathsf{ev}_N\left(s^*\left(\nu_{(2)}\right)\right) &= \sum \nu_{(1)}s^*\left(\nu_{(2)}\right)\left(N\right) = \sum \nu_{(1)}\nu_{(2)}\left(s\left(N\right)\right) \\
&= s\left(N\right)\rightharpoonup\nu = n\rightharpoonup\nu = \varepsilon,
\end{aligned}$$

which means that ev_N is a Frobenius homomorphism for H^*. Therefore, we may apply Lemma 4.20 to conclude that $\int_{H^*}^l$ has to be free of rank one. $\square$

From the proof of Theorem 4.22 it is clear that if B is a bialgebra which is Frobenius as an algebra and such that $\nu \in \int_{B^*}^l$, then an antipode for B is given explicitly by $s(a) := \sum n_{(1)}\nu\left(an_{(2)}\right)$ and its inverse by $s^{-1}(a) = \sum N_{(1)}\nu\left(N_{(2)}a\right)$, where n is a left norm and N a right norm. Moreover, $\int_{B^*}^l$ is the free $\Bbbk$-module generated by ν. Conversely, assume that we know that B is a Frobenius Hopf algebra and that we know a (free) generator φ of $\int_{B^*}^l$. We want to answer to the following questions:

- How do we find a generator for $\int_B^l$?
- Can we explicit describe a Frobenius homomorphism and a Casimir element for B?
- What is the inverse of the isomorphism $\psi : B \to B^* : b \longmapsto (b \rightharpoonup \varphi)$?

Proposition 4.23. *Let H be a Frobenius Hopf algebra with (free) generator φ of $\int_{B^*}^l$. Then $\psi : B \to B^* : b \longmapsto (b \rightharpoonup \varphi)$ is a left H-linear isomorphism, $t := \psi^{-1}(\varepsilon)$ is a free generator of $\int_B^l$ and*

$$\psi^{-1} : B^* \to B : f \longmapsto \sum f\left(S^{-1}\left(t_{(1)}\right)\right) t_{(2)}.$$

Moreover, $\left(\varphi, \sum t_{(2)} \otimes S^{-1}\left(t_{(1)}\right)\right)$ is a Frobenius system for H.

Proof. If B is a finitely generated and projective Hopf algebra and $\varphi \in \int_{B^*}^l$ is a free generator, we have a right B-linear isomorphism $\gamma : B \to B^* : b \longmapsto (S(b) \rightharpoonup \varphi)$ which induces a left B-linear isomorphism $\psi : B \to B^* : b \longmapsto \gamma(S^{-1}(b))$. We already know from Lemma 4.20 and Remark 4.21 that $t := \psi^{-1}(\varepsilon)$ is a free generator of $\int_B^l$. Consider the Casimir element $\sum t_{(2)} \otimes S^{-1}\left(t_{(1)}\right)$ and compute

$$\sum \varphi\left(at_{(2)}\right) S^{-1}\left(t_{(1)}\right) = \sum \varphi\left(t_{(2)}\right) S^{-1}\left(t_{(1)}\right) a = S^{-1}\left(\sum t_{(1)}\varphi\left(t_{(2)}\right)\right) a = \varphi(t)\, S^{-1}(1)\, a = a.$$

Thus, in light of (3) $\Longrightarrow$ (1) from Proposition 4.13 we conclude that a Frobenius homomorphism for B is given by φ and an associated Casimir element by $\sum t_{(2)} \otimes S^{-1}\left(t_{(1)}\right)$. $\square$

Theorem 4.24. *For a Hopf algebra H, the following assertions are equivalent*

(a) *H is Frobenius over $\Bbbk$;*
(b) *H is finitely generated and projective and H^* is Frobenius over $\Bbbk$;*
(c) *H is finitely generated and projective and $\int_H^l$ is free of rank one;*
(d) *H is finitely generated and projective and $\int_H^r$ is free of rank one;*
(e) *H is finitely generated and projective and $\int_{H^*}^l$ is free of rank one;*
(f) *H is finitely generated and projective and $\int_{H^*}^r$ is free of rank one.*

Proof. The implication from (a) to (c) is the content of Lemma 4.20. The implication from (a) to (b) is already contained in the second part of the proof of Theorem 4.22. The equivalence between (a) and (e) is the content of Theorem 4.22. By recalling that $H \cong H^{**}$ and that $\int_H^l \cong \int_{H^{**}}^l$, $\int_H^r \cong \int_{H^{**}}^r$, we see that the equivalence between (b) and (c) is again Theorem 4.22 applied to H^*. Since the antipode is invertible and since it maps left integrals to right integrals (and conversely) by Remark 4.6, we have that (c) $\iff$ (d) and (e) $\iff$ (f). Finally, the implication from (b) to (e) is the same as the implication from (a) to (c) but applied to H^*. $\square$

Remark 4.25. It may sound redundant, but the additional hypothesis that H is finitely generated and projective in point (b) is necessary. Being H^* Frobenius tells us that H^* is finitely generated and projective, but this does not imply that H is finitely generated and projective in general.

Exercise 4.26. Prove that $\mathsf{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) = 0$ (whence it is finitely generated and projective).

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Département de Mathématique, Université Libre de Bruxelles, Boulevard du Triomphe, B-1050 Brussels, Belgium.
URL: `sites.google.com/view/paolo-saracco`
E-mail address: `paolo.saracco@ulb.ac.be`